

WALDHAUSEN ADDITIVITY: CLASSICAL AND QUASICATEGORICAL

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ABSTRACT. We give a short proof of classical Waldhausen Additivity, and then prove Waldhausen Additivity for an ∞ -version of Waldhausen K -theory. Namely, we prove that Waldhausen K -theory sends a split-exact sequence of Waldhausen quasicategories $\mathcal{A} \rightarrow \mathcal{E} \rightarrow \mathcal{B}$ to a stable equivalence of spectra $K(\mathcal{E}) \rightarrow K(\mathcal{A}) \vee K(\mathcal{B})$ under a few mild hypotheses. For example, each cofiber sequence in \mathcal{A} of the form $A_0 \rightarrow A_1 \rightarrow *$ is required to have the first map an equivalence. Model structures, presentability, and stability are not needed. In an effort to make the article self-contained, we provide many details in our proofs, recall all the prerequisites from the theory of quasicategories, and prove some of those as well. For instance, we develop the expected facts about (weak) adjunctions between quasicategories and (weak) adjunctions between simplicial categories.

Key words: quasicategory, ∞ -category, K -theory, Additivity, Waldhausen quasicategory,

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0. INTRODUCTION AND STATEMENT OF RESULTS

The Additivity Theorems of Quillen [26, §3, Theorem 2] and Waldhausen [32, Theorem 1.4.2] are fundamental theorems in K -theory; many other results follow from them [16, 29]. Quillen’s Theorem A is the key ingredient in most proofs of Additivity. In this paper, we present a short proof of Additivity using Theorem \hat{A}^* of Jones–Kim–Mhooon–Santhanam–Walker–Grayson [18], which is a simplicial version of Quillen’s Theorem A designed for products. Another key ingredient for our paper is a variation on a homotopy found in McCarthy’s proof [25].

The advantage of our formulation is that it also works for quasicategories. We use the same method to prove Additivity for Lurie’s quasicategorical version of Waldhausen K -theory defined in [22, 1.2.2.5, page 40]. Although the proof is entirely simplicial, we were very surprised to see the important role of categorical aspects of quasicategories in the various formulations of Additivity. Indeed, much of this paper relies upon quasicategorical versions of basic results of category theory, as developed by Joyal [20], rather than arguments from abstract homotopy theory.

Waldhausen Additivity, in its most specific form, says the following. Let \mathcal{C} be a Waldhausen category. This is a category equipped with a distinguished zero object, a subcategory of cofibrations, and subcategory of weak equivalences satisfying various axioms, most notably: pushouts of cofibrations exist, are cofibrations, and are invariant up to weak equivalence (=gluing lemma). If $S_2\mathcal{C}$ denotes the category of cofiber sequences $A \twoheadrightarrow C \twoheadrightarrow B$ in \mathcal{C} , and $s, q: S_2\mathcal{C} \rightarrow \mathcal{C}$ are the “sub-object” and “quotient” functors which return A and B respectively, then Waldhausen Additivity [32, Theorem 1.4.2] says that the map of simplicial objects in \mathbf{Cat}

$$(0.1) \quad wS_\bullet(s, q): wS_\bullet S_2\mathcal{C} \rightarrow wS_\bullet\mathcal{C} \times wS_\bullet\mathcal{C}$$

is a weak equivalence, that is, its level-wise nerve is a diagonal weak equivalence of bisimplicial sets. See Definition 1.1 and equation (1.2) for a recollection of Waldhausen’s S_\bullet construction.

More generally, if \mathcal{A} and \mathcal{B} are sub Waldhausen categories of \mathcal{C} , in place of $S_2\mathcal{C}$ we may consider the category $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$, which consists of cofiber sequences $A \twoheadrightarrow C \twoheadrightarrow B$ in \mathcal{C} with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Then

$$(0.2) \quad wS_\bullet(s, q): wS_\bullet\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \rightarrow wS_\bullet\mathcal{A} \times wS_\bullet\mathcal{B}$$

is a weak equivalence. Waldhausen proved in [32, Proposition 1.3.2] that this apparently more general statement is equivalent to the claim that (0.1) is a weak equivalence. In Section 1 we prove directly that (0.2) is a weak equivalence.

Still more generally, if

$$(0.3) \quad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{f} & \mathcal{B} \\ & \nwarrow j & & \nearrow g & \\ & & & & \end{array}$$

is a split-exact sequence of Waldhausen categories (Definitions 2.1 and 2.3) satisfying two natural conditions, then the functors j and f induce a weak equivalence

$$(0.4) \quad wS_{\bullet}(j, f): wS_{\bullet}\mathcal{E} \longrightarrow wS_{\bullet}\mathcal{A} \times wS_{\bullet}\mathcal{B}$$

and a stable equivalence of K -theory spectra

$$(0.5) \quad K(j, f): K(\mathcal{E}) \longrightarrow K(\mathcal{A}) \vee K(\mathcal{B}).$$

We prove this in Section 2 as a consequence of the weak equivalence (0.2). Indeed, Proposition 2.10 tells us that any split-exact sequence (0.3) satisfying these natural conditions is Waldhausen equivalent (Definition 6.1) to a split-exact sequence of the form

$$\mathcal{A}' \rightarrow \mathcal{E}(\mathcal{A}', \mathcal{C}', \mathcal{B}') \rightarrow \mathcal{B}'$$

as in (0.2).

The main goal of this paper is to directly prove quasicategorical versions of the foregoing results. Our strategy is to first prove these classical results in such a way that the proofs carry over to the quasicategorical context. In an effort to make this paper self-contained, we recall in Section 3 all of the notions and results we need from the theory of quasicategories. We prove some of the elementary results as well, such as the fact that a functor between quasicategories is an equivalence if and only if it is (weakly) fully faithful and essentially surjective, or the fact that a right adjoint is a fully faithful if and only if the counit is a natural equivalence. Pushouts in a quasicategory are also described.

In Section 4 we prove the quasicategorical version of the weak equivalence (0.2). Section 5 deals with the analogues of (0.4) and (0.5). In Section 6, the Appendix, we characterize Waldhausen equivalences as those exact equivalences which reflect weak equivalences and cofibrations, and show that wS_{\bullet} of a Waldhausen equivalence is a weak equivalence of simplicial objects in \mathbf{Cat} . The analogue for S_{\bullet}^{∞} is included.

Two articles recently posted on the arXiv concern Additivity for quasicategorical versions of K -theory: the article [2] of Blumberg–Gepner–Tabuada and the article [1] of Barwick. The work of Blumberg–Gepner–Tabuada, like ours, uses a quasicategorical S_{\bullet} construction, however, unlike the present article, they proceed using Morita theory, spectrally enriched categories, strictification, and classical Waldhausen Additivity. The present article remains entirely in the framework of quasicategories, except for a very brief detour through simplicial categories

to discuss universal properties of weak adjunctions. Also, in [2] every map is considered a cofibration because they work with stable quasicalegories, whereas in the present article we propose a quasicalegorical analogue of Waldhausen category in which the class of cofibrations can be more general, see Definition 4.1.

The article [1] of Barwick also remains in the framework of quasicalegories, although he does not use a quasicalegorical S_\bullet construction. Instead, he characterizes additive theories, and defines algebraic K -theory as an additive theory with a universal property (the linearization of the functor “maximal Kan subcomplex”). He then proves quasicalegorical versions of the basic theorems of algebraic K -theory, and then proves that this universal definition extends the classical one. Barwick allows the class of cofibrations to be general, though he uses a slightly different notion of Waldhausen quasicalegory than in the present paper.

Both papers [1] and [2] are concerned with universal characterizations of algebraic K -theory via Additivity, whereas the present paper is only concerned with a direct proof of Additivity. Quasicalegorical variants of Waldhausen’s Approximation Theorem are treated in [1], [2], and [11].

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1. PROOF OF ADDITIVITY IN CLASSICAL CASE

We recall Waldhausen’s S_\bullet construction [32], and combine the proofs of McCarthy [25] and Jones–Kim–Mhoon–Santhanam–Walker–Grayson [18] to quickly prove Waldhausen Additivity via Theorem \hat{A}^* .

Let \mathcal{C} be a *Waldhausen category*, that is, a category equipped with a distinguished zero object $*$, a subcategory $co\mathcal{C}$ of *cofibrations*, and a subcategory $w\mathcal{C}$ of *weak equivalences* satisfying Waldhausen’s axioms [32, pages 320 and 326]. These require that all isomorphisms are both cofibrations and weak equivalences, for each object A the unique map $* \rightarrow A$ is a cofibration, the pushout of each cofibration exists and is again a cofibration, and finally, the gluing lemma holds. Cofibrations are always indicated with a feathered arrow \rightarrowtail .

Every model category contains an example of a Waldhausen category. Namely, if \mathcal{M} is a model category, then its full subcategory \mathcal{M}_c on its cofibrant objects is a Waldhausen category. For a proof of the gluing lemma in this generality, see Goerss–Jardine [15, pages 122–123 and 127] or Hovey [17, Lemma 5.2.6].

Let \mathcal{B} and \mathcal{C} be Waldhausen categories. A functor $\mathcal{B} \rightarrow \mathcal{C}$ is *exact* if it takes $*$ to $*$, maps cofibrations to cofibrations, maps weak equivalences to weak equivalences, and maps each pushout along a cofibration to a pushout along a cofibration. A *sub Waldhausen category* of \mathcal{C} is a subcategory \mathcal{C}' with a Waldhausen structure such that the inclusion functor is exact *and additionally* a morphism in the subcategory \mathcal{C}' is a cofibration in the subcategory \mathcal{C}' if it is a cofibration in the larger category \mathcal{C} and the quotient is in \mathcal{C}' up to isomorphism.

The weak equivalences of a Waldhausen subcategory \mathcal{C}' may not include fully faithfully into the weak equivalences of the larger Waldhausen category \mathcal{C} .

We next want to recall the S_\bullet construction, a main ingredient in the Additivity Theorem. Let $\text{Ar}[n]$ be the *category of arrows* in $[n]$. It is the partially ordered set with elements (i, j) such that $0 \leq i \leq j \leq n$, and with the order $(i, j) \leq (i', j')$ whenever $i \leq i'$ and $j \leq j'$.

Definition 1.1 (Waldhausen’s S_\bullet construction, Section 1.3 of [32]). Let \mathcal{C} be Waldhausen category. An *object of the category $S_n\mathcal{C}$* is a functor $A: \text{Ar}[n] \rightarrow \mathcal{C}$ such that

- (i) For each $i \in [n]$, $A(i, i)$ is the distinguished zero object of \mathcal{C} ,
- (ii) For each $i \leq j \leq k$, the morphism $A_{i,j} \rightarrow A_{i,k}$ is a cofibration,
- (iii) For each $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} A(i, j) & \twoheadrightarrow & A(i, k) \\ \downarrow & & \downarrow \\ A(j, j) & \twoheadrightarrow & A(j, k) \end{array}$$

is a pushout square in \mathcal{C} .

The *morphisms of the category $S_n\mathcal{C}$* are natural transformations. The assignment $[n] \mapsto \text{Ar}[n] \rightarrow S_n\mathcal{C}$ is a functor $\Delta^{\text{op}} \rightarrow \mathbf{Cat}$, so that $S_\bullet\mathcal{C}$ is a simplicial object in \mathbf{Cat} .

The objects of $S_n\mathcal{C}$ are sequences of cofibrations in \mathcal{C}

$$(1.2) \quad * \twoheadrightarrow A_{0,1} \twoheadrightarrow A_{0,2} \twoheadrightarrow \cdots \twoheadrightarrow A_{0,n}$$

with a choice of quotient $A_{i,j} = A_{0,j}/A_{0,i}$ for each $i \leq j$. For $i > 0$, the face map $d_i: S_n\mathcal{C} \rightarrow S_{n-1}\mathcal{C}$ composes the two morphisms $A_{0,i-1} \rightarrow A_{0,i} \rightarrow A_{0,i+1}$ in (1.2), which corresponds to removing object column i and object row i in the grid A . The face map $d_0: S_n\mathcal{C} \rightarrow S_{n-1}\mathcal{C}$ removes object column 0 and object row 0 in the grid A , which means

to quotient (1.2) by $A_{0,1}$ and to remove the first $*$ to make

$$* \twoheadrightarrow A_{1,2} \twoheadrightarrow A_{1,3} \twoheadrightarrow \cdots \twoheadrightarrow A_{1,n}.$$

For $i \geq 0$, the degeneracy map $s_i: S_n\mathcal{C} \rightarrow S_{n+1}\mathcal{C}$ replaces $A_{0,i}$ in (1.2) by the identity morphism $A_{0,i} \rightarrow A_{0,i}$. This corresponds to inserting a column of trivial morphisms and a row of trivial morphisms in the grid A in object column i and object row i .

The category $S_n\mathcal{C}$ is a Waldhausen category. A natural transformation $f: A \rightarrow B$ in $S_n\mathcal{C}$ is a *cofibration in $S_n\mathcal{C}$* if each pushout morphism

$$A_{0,j} \cup_{A_{0,j-1}} B_{0,j-1} \rightarrow B_{0,j}$$

is a cofibration in \mathcal{C} for each $1 \leq j \leq n$. This implies, but is not equivalent to, the requirement that each $f_{i,j}: A_{i,j} \rightarrow B_{i,j}$ be a cofibration in \mathcal{C} , see Rognes' notes [27, Lemma 8.3.12]. A natural transformation $f: A \rightarrow B$ is a *weak equivalence in $S_n\mathcal{C}$* if $f_{0,j}: A_{0,j} \rightarrow B_{0,j}$ is a weak equivalence in \mathcal{C} . This is equivalent to requiring each $f_{i,j}$ to be a weak equivalence in \mathcal{C} , see [27, 8.3.15]. Each structure map of $S_\bullet\mathcal{C}$ is exact, so $S_\bullet\mathcal{C}$ is a simplicial object in Waldhausen categories. We denote the simplicial set of the object sets of $S_\bullet\mathcal{C}$ by $\mathfrak{s}_\bullet\mathcal{C}$, that is, $\mathfrak{s}_n\mathcal{C} = \text{Obj } S_n\mathcal{C}$.

The objects of $S_2\mathcal{C}$ are *cofiber sequences* $A \twoheadrightarrow C \twoheadrightarrow B$ in \mathcal{C} , that is, pushouts of the form

$$\begin{array}{ccc} A & \twoheadrightarrow & C \\ \downarrow & & \downarrow \\ * & \twoheadrightarrow & B, \end{array}$$

where the arrow \twoheadrightarrow always indicates the projection to the quotient. The “subobject” and “quotient” functors $s, q: S_2\mathcal{C} \rightarrow \mathcal{C}$, which assign to the cofiber sequence $A \twoheadrightarrow C \twoheadrightarrow B$ the objects A and B respectively, are exact.

The first version of Waldhausen Additivity says that the Waldhausen category of cofiber sequences in \mathcal{C} is weakly equivalent to $\mathcal{C} \times \mathcal{C}$ after taking weak equivalences in the S_\bullet construction.

Theorem 1.3 (Waldhausen Additivity Theorem 1.4.2 of [32]). *Let \mathcal{C} be a Waldhausen category. Then the “subobject” and “quotient” functors induce a weak equivalence of simplicial objects in \mathbf{Cat}*

$$wS_\bullet(s, q): wS_\bullet S_2\mathcal{C} \rightarrow wS_\bullet\mathcal{C} \times wS_\bullet\mathcal{C}.$$

That is, the diagonal of the level-wise nerve is a weak equivalence of simplicial sets.

Waldhausen deduces the Additivity Theorem from a weak equivalence on the level of object simplicial sets: his Lemma 1.4.3 says that $\mathfrak{s}_\bullet(s, q): \mathfrak{s}_\bullet S_2\mathcal{C} \rightarrow \mathfrak{s}_\bullet\mathcal{C} \times \mathfrak{s}_\bullet\mathcal{C}$ is a weak equivalence of simplicial sets. The proof of Lemma 1.4.3 for object simplicial sets included an application of Quillen's Theorem B and a certain technical Sublemma. In [25],

McCarthy gave a new proof of Waldhausen’s Lemma 1.4.3 for object simplicial sets in the spirit of Quillen’s Theorem A without using Waldhausen’s Sublemma. Later in [18], Jones–Kim–Mhoon–Santhanam–Walker–Grayson proved Theorem \hat{A}^* (recalled below in Theorem 1.8), which generally allows one to convert Theorem B style proofs into Theorem A style proofs. However, their proof of the weak equivalence on the level of object simplicial sets in [18, Lemma 2.4] uses results of Waldhausen’s technical Sublemma.

In the present paper, we use Theorem \hat{A}^* of [18], in combination with a slight variation of the simplicial homotopy for $\Gamma \simeq \text{Id}_{S_\bullet F|\mathcal{C}^2(-,[n])}$ in [25], to give a self-contained proof of a more general weak equivalence on the level of object simplicial sets \mathfrak{s}_\bullet , *without* Waldhausen’s Sublemma. For this more general weak equivalence in Lemma 1.9, the subobject and the quotient in the cofiber sequences are in sub Waldhausen categories \mathcal{A} and \mathcal{B} . The Additivity Theorem for wS_\bullet for such cofiber sequences in Theorem 1.15 follows. In Theorem 2.19 we prove Waldhausen Additivity for split-exact sequences of Waldhausen categories, and we do the spectral version in Theorem 2.20.

We follow Waldhausen’s notations for the category of cofiber sequences in \mathcal{C} with the subobject and the quotient in sub Waldhausen categories \mathcal{A} and \mathcal{B} .

Notation 1.4. Let \mathcal{C} be a Waldhausen category and \mathcal{A} and \mathcal{B} sub Waldhausen categories. We denote by $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ the category of cofiber sequences in \mathcal{C} of the form

$$A \twoheadrightarrow C \twoheadrightarrow B$$

with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A morphism consists of three morphisms in \mathcal{A} , \mathcal{C} , and \mathcal{B} respectively, making the two relevant squares commute.

Proposition 1.5 (Waldhausen [32]). *Let \mathcal{C} be a Waldhausen category and \mathcal{A} and \mathcal{B} sub Waldhausen categories. Then $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is a Waldhausen category with levelwise weak equivalences and cofibrations the morphisms such that the induced map $C_1 \cup_{A_1} A_2 \rightarrow C_2$ is a cofibration. The projections to \mathcal{A} , \mathcal{C} , and \mathcal{B} are exact. In particular, the three components of a cofibration are also cofibrations.*

Before proving Lemma 1.9, we recall Theorem \hat{A}^* and left fibers in simplicial sets. Theorem \hat{A}^* is a simplicial version of Quillen’s Theorem A adapted for the situation of products.

Definition 1.6 (Left fiber in simplicial sets, pages 336–337 of [32]). For any simplicial set map $f: X \rightarrow Y$ and any $y \in Y_m$, the *left fiber*

$f/(m, y)$ is defined as the following pullback in simplicial sets.

$$\begin{array}{ccc} f/(m, y) & \xrightarrow{\pi} & X \\ \downarrow & \text{pullback} & \downarrow f \\ \Delta[m] & \xrightarrow{y} & Y \end{array}$$

By the universal property of the pullback and several applications of the Yoneda Lemma, an n -simplex $\Delta[n] \rightarrow f/(m, y)$ is the same as an n -simplex x of X together with a morphism $\alpha: [n] \rightarrow [m]$ in Δ such that $\alpha^*y = f(x)$.

Theorem 1.7 (Simplicial Version of Quillen’s Theorem A, Lemma 1.4.A of [32]). *If $f/(m, y)$ is contractible for every (m, y) , then f is a weak equivalence.*

Theorem 1.8 (Theorem \hat{A}^* of Jones–Kim–Mhoon–Santhanam–Walker–Grayson¹ in [18]). *Let $(f, g): X \rightarrow Y \times T$ be a map of simplicial sets. If the composite*

$$f/(m, y) \xrightarrow{\pi} X \xrightarrow{g} T$$

is a weak equivalence for all $m \geq 0$ and all $y \in Y_m$, then (f, g) is a weak equivalence of simplicial sets.

The following has Waldhausen’s [32, Lemma 1.4.3] as the special case $\mathcal{A} = \mathcal{B} = \mathcal{C}$.

Lemma 1.9 (Waldhausen Additivity for Object Simplicial Sets). *Let \mathcal{C} be a Waldhausen category and \mathcal{A} and \mathcal{B} sub Waldhausen categories. Then the “subobject” and “quotient” functors induce a weak equivalence of simplicial sets*

$$\mathfrak{s}_\bullet(s, q): \mathfrak{s}_\bullet\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \rightarrow \mathfrak{s}_\bullet\mathcal{A} \times \mathfrak{s}_\bullet\mathcal{B} .$$

Proof. We begin this proof as in the notes of Rognes [27, Lemma 8.5.11]. Let (f, g) be $(\mathfrak{s}_\bullet s, \mathfrak{s}_\bullet q): \mathfrak{s}_\bullet\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \rightarrow \mathfrak{s}_\bullet\mathcal{A} \times \mathfrak{s}_\bullet\mathcal{B}$, and for any fixed $A' \in \mathfrak{s}_m\mathcal{A}$ denote by r the composite

$$r: f/(m, A') \xrightarrow{\pi} \mathfrak{s}_\bullet\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \xrightarrow{g} \mathfrak{s}_\bullet\mathcal{B} .$$

¹In the statement of their theorem in [18], Jones–Kim–Mhoon–Santhanam–Walker–Grayson use the term “homotopy equivalence” rather than “weak equivalence”. In the current paper we prefer to use the term “weak equivalence” for maps of simplicial sets which induce a homotopy equivalence after geometric realization, in order to distinguish them from simplicial homotopy equivalences.

An n -simplex e in $f/(m, A')$ is a diagram in \mathcal{C} of the form

$$(1.10) \quad \begin{array}{ccccccc} * & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots \longrightarrow & A_n \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ * & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & \cdots \longrightarrow & C_n \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ * & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \longrightarrow & B_n \end{array} ,$$

with top row in \mathcal{A} , middle row in \mathcal{C} , and bottom row in \mathcal{B} , and with chosen quotients, together with a morphism $\alpha: [n] \rightarrow [m]$ in Δ such that $\alpha^* A'$ is the top row A . Here we are using the facts that $\mathfrak{s}_n S_2 \mathcal{C} \cong \mathfrak{s}_2 S_n \mathcal{C}$ and an object of $S_n \mathcal{C}$ is a sequence of cofibrations, as in equation (1.2), with chosen quotients. The rows of diagram (1.10) are sequences of cofibrations and the columns are cofiber sequences, and we set $A_0 = C_0 = B_0 = *$ as a convention.

The composite r maps the n -simplex e above to $B \in \mathfrak{s}_n \mathcal{B}$, the bottom row of diagram (1.10). We prove that r is a simplicial deformation retraction and therefore a weak equivalence of simplicial sets. That is, we display a map $\iota: \mathfrak{s}_\bullet \mathcal{B} \rightarrow f/(m, A')$ such that $r \circ \iota = \text{Id}_{\mathfrak{s}_\bullet \mathcal{B}}$ and a simplicial homotopy $h: \iota \circ r \simeq \text{Id}_{f/(m, A')}$. Then we are finished by Theorem \hat{A}^* .

For $B \in \mathfrak{s}_n \mathcal{B}$, we define $\iota(B)$ to be the diagram

$$(1.11) \quad \begin{array}{ccccccc} * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & \cdots \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ * & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \longrightarrow & B_n \\ \parallel & & \parallel & & \parallel & & & \parallel \\ * & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots \longrightarrow & B_n \end{array}$$

equipped with the morphism $\beta: [n] \rightarrow [m]$ which maps each $j \in [n]$ to m . Strictly speaking, the bottom and middle rows also have the chosen quotients of B , so are functors $\text{Ar}[n] \rightarrow \mathcal{B}$, but we do not include them in the notation for readability. The top row has chosen quotients all $*$. Since β factors as $[n] \rightarrow [0] \rightarrow [m]$, we have $\beta^* A'$ is equal to the top row of (1.11).

What we have said in this proof so far is in [27, Lemma 8.5.11] for the special case $\mathcal{A} = \mathcal{B} = \mathcal{C}$. The following simplicial homotopy $h: \iota \circ r \simeq \text{Id}_{f/(m, A')}$ is a variation on a homotopy of McCarthy [25], and is the new aspect of this proof. We use a more modern reformulation of the notion of simplicial homotopy, as in [27, Lemma 6.4.8]. Throughout

this proof, we use a selected functorial choice² of pushouts in \mathcal{C} along cofibrations.

For each $n \geq 0$ and $0 \leq j \leq n+1$, define functions

$$h_n^j: (f/(m, A'))_n \rightarrow (f/(m, A'))_n$$

by $h_n^0 = (\iota \circ r)_n$ and $h_n^{n+1} = (\text{Id}_{f/(m, A')})_n$. For $1 \leq j \leq n$ and an n -simplex e as in (1.10), we define

$$h_n^j(e) := \begin{array}{ccccccc} * & \xrightarrow{\quad} & A_1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & A_{j-1} & \xrightarrow{\quad} & A_n & \xlongequal{\quad} & A_n & \xlongequal{\quad} & \cdots & \xlongequal{\quad} & A_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ * & \xrightarrow{\quad} & C_1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & C_{j-1} & \xrightarrow{\quad} & X_j & \xrightarrow{\quad} & X_{j+1} & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & X_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ * & \xrightarrow{\quad} & B_1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & B_{j-1} & \xrightarrow{\quad} & B_j & \xrightarrow{\quad} & B_{j+1} & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & B_n \end{array}$$

Here X_k is the (functorially chosen) pushout $C_k \cup_{A_k} A_n$ for $j \leq k \leq n$. The chosen quotients for the rows are as follows. The quotients for the top and bottom rows are obtained from the chosen quotients for the top and bottom rows of (1.10). In the middle row of $h_n^j(e)$, the quotients C_t/C_s for $1 \leq s < t \leq j-1$ are the same as the respective ones from the middle row of (1.10). For the quotients X_k/C_s with $1 \leq s < j \leq k$ we *choose* them to be the (functorially chosen) pushout

$$(1.12) \quad X_k/C_s := (C_k/C_s) \bigcup_{A_k/A_s} (A_n/A_s).$$

Observe that this choice for X_k/C_s really is a quotient of X_k by C_s because each side of (1.12) calculates the colimit of the diagram

$$\begin{array}{ccccc} * & \xlongequal{\quad} & * & \xlongequal{\quad} & * \\ \uparrow & & \uparrow & & \uparrow \\ C_s & \xleftarrow{\quad} & A_s & \xlongequal{\quad} & A_s \\ \downarrow & & \downarrow & & \downarrow \\ C_k & \xleftarrow{\quad} & A_k & \xrightarrow{\quad} & A_n \end{array}$$

in a different way. For $j \leq k < \ell$, we choose the quotient X_ℓ/X_k to be

$$(1.13) \quad X_\ell/X_k := B_\ell/B_k,$$

²Let \mathcal{I} be the category $\bullet \leftarrow \bullet \rightarrow \bullet$ and \mathcal{J} the category consisting of the free standing commutative square. Then the restriction functor $\mathbf{Cat}(\mathcal{J}, \mathcal{C}) \rightarrow \mathbf{Cat}(\mathcal{I}, \mathcal{C})$ restricts to an equivalence of categories between the full subcategory of $\mathbf{Cat}(\mathcal{J}, \mathcal{C})$ spanned by the pushouts with first leg a cofibration and the full subcategory of $\mathbf{Cat}(\mathcal{I}, \mathcal{C})$ on diagrams with the first leg a cofibration. An inverse equivalence to this equivalence is a functorial choice of pushouts in \mathcal{C} along cofibrations.

which is already chosen in B . We see that this choice really is a quotient of X_ℓ by X_k by computing the colimit of the diagram

$$\begin{array}{ccccc}
 * & \xlongequal{\quad} & * & \xlongequal{\quad} & * \\
 \uparrow & & \uparrow & & \uparrow \\
 C_k & \xleftarrow{\quad} & A_k & \xrightarrow{\quad} & A_n \\
 \downarrow & & \downarrow & & \downarrow \\
 C_\ell & \xleftarrow{\quad} & A_\ell & \xrightarrow{\quad} & A_n
 \end{array}$$

in two ways.

We proceed to comment on the required identities for the simplicial homotopy. Recall the structure maps d_i and s_i after equation (1.2). We first treat all the identities involving d_0 .

For $j = 0$, we have $d_0(h_n^0(e)) = h_{n-1}^0(d_0(e))$ because both sides are equal to

$$\begin{array}{ccccccc}
 * & \xrightarrow{\quad} & * & \xrightarrow{\quad} & * & \xrightarrow{\quad} & \cdots \xrightarrow{\quad} * \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 * & \xrightarrow{\quad} & B_2/B_1 & \xrightarrow{\quad} & B_3/B_1 & \xrightarrow{\quad} & \cdots \xrightarrow{\quad} B_n/B_1 \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 * & \xrightarrow{\quad} & B_2/B_1 & \xrightarrow{\quad} & B_3/B_1 & \xrightarrow{\quad} & \cdots \xrightarrow{\quad} B_n/B_1
 \end{array}$$

For $j = 1$, we have $d_0(h_n^1(e)) = h_{n-1}^{1-1}(d_0(e))$ as follows. The top row of $h_n^1(e)$ is $* \rightharpoonup A_n = A_n = \cdots = A_n$, which becomes $* \rightharpoonup * \rightharpoonup \cdots \rightharpoonup *$ after modding out by A_n and contracting the initial $*$, as d_0 does. The middle row of $h_n^1(e)$ is $* \rightharpoonup X_1 \rightharpoonup \cdots \rightharpoonup X_n$, and by (1.13) the middle row of $d_0(h_n^1(e))$ is

$$* \rightharpoonup B_2/B_1 \rightharpoonup B_3/B_1 \rightharpoonup \cdots \rightharpoonup B_n/B_1,$$

which is also the bottom row of $d_0(h_n^1(e))$ by definition of d_0 . Thus we have $d_0(h_n^1(e)) = (\iota \circ r)_n(d_0 e) = h_{n-1}^0(d_0(e))$.

For $2 \leq j \leq n$, we similarly have $d_0(h_n^j(e)) = h_{n-1}^{j-1}(d_0(e))$ as follows. The top and bottom rows are clear, while the equation holds for the middle row precisely because of definition (1.12) with $s = 1$.

For $j = n + 1$, we have $d_0(h_n^{n+1}(e)) = h_{n-1}^{(n+1)-1}(d_0(e))$ because both h_n^{n+1} and h_{n-1}^n are the relevant identity functions.

For $n \geq 1$, and $i \geq 1$, the equations

$$d_i(h_n^j(e)) = \begin{cases} h_{n-1}^{j-1}(d_i(e)) & \text{for } 1 \leq i < j \leq n + 1 \\ h_{n-1}^j(d_i(e)) & \text{for } j \leq i \leq n \end{cases}$$

are not difficult to see. For instance, for $j = i \geq 1$, composing away the middle column in the subdiagram of $h_n^j(e)$

$$\begin{array}{ccccc}
 A_{j-1} & \xrightarrow{\quad} & A_n & \xlongequal{\quad} & A_n \\
 \downarrow & & \downarrow & & \downarrow \\
 C_{j-1} & \xrightarrow{\quad} & X_j & \xrightarrow{\quad} & X_{j+1} \\
 \downarrow & & \downarrow & & \downarrow \\
 B_{j-1} & \xrightarrow{\quad} & B_j & \xlongequal{\quad} & B_{j+1}
 \end{array}$$

gives us the relevant subdiagram of $h_{n-1}^j(d_j(e))$.

For $n \geq 0$, the equations

$$s_i(h_n^j(e)) = \begin{cases} h_{n+1}^{j+1}(s_i(e)) & \text{for } 0 \leq i < j \\ h_{n+1}^j(s_i(e)) & \text{for } j \leq i \leq n \end{cases}$$

are not difficult to see. For $j = 0$, we have $h_n^0 = (\iota \circ r)_n$ and $h_{n+1}^0 = (\iota \circ r)_{n+1}$, which commute with s_i because $\iota \circ r$ is a map of simplicial sets. For $1 \leq j \leq n$, we look at the diagram defining $h_j(e)$ and notice that the two cases correspond to inserting an identity before column j , or at column j or beyond. \square

For readers who prefer the classical formulation of a simplicial homotopy as in the book of May [24], we also present the simplicial homotopy in those terms.

Proof. Simplicial Homotopy $h: \iota \circ r \simeq Id_{f/(m,A')}$ in Classical Formulation

For each $n \geq 0$ and $0 \leq j \leq n$, define functions $h_j: (f/(m, A'))_n \rightarrow (f/(m, A'))_{n+1}$ on an n -simplex e as in (1.10) by

$$h_j(e) := \begin{array}{ccccccccccccccc}
 * & \xrightarrow{\quad} & A_1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & A_j & \xrightarrow{\quad} & A_n & \xlongequal{\quad} & A_n & \xlongequal{\quad} & \cdots & \xlongequal{\quad} & A_n \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
 * & \xrightarrow{\quad} & C_1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & C_j & \xrightarrow{\quad} & X_j & \xrightarrow{\quad} & X_{j+1} & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & X_n \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
 * & \xrightarrow{\quad} & B_1 & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & B_j & \xlongequal{\quad} & B_j & \xrightarrow{\quad} & B_{j+1} & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & B_n
 \end{array}$$

Here X_k is the pushout $C_k \cup_{A_k} A_n$ for $j \leq k \leq n$, where we have made a global functorial choice of pushout. The chosen quotients for the rows are as follows. The quotients for the top and bottom rows are obtained from the chosen quotients for the top and bottom rows of (1.10). In the middle row of $h_j(e)$, the quotients C_t/C_s for $1 \leq s \leq t \leq j$ are the same ones from the middle row of (1.10). For the quotients X_k/C_s

with $1 \leq s \leq j \leq k$ we *choose* them to be the functorial choice of the pushout mentioned above.

$$(1.14) \quad X_k/C_s := (C_k/C_s) \bigcup_{A_k/A_s} (A_n/A_s)$$

Observe that this choice for X_k/C_s really is a quotient of X_k by C_s because each side of (1.14) calculates the colimit of the diagram

$$\begin{array}{ccccc} * & \xlongequal{\quad} & * & \xlongequal{\quad} & * \\ \uparrow & & \uparrow & & \uparrow \\ C_s & \xleftarrow{\quad} & A_s & \xlongequal{\quad} & A_s \\ \downarrow & & \downarrow & & \downarrow \\ C_k & \xleftarrow{\quad} & A_k & \xrightarrow{\quad} & A_n \end{array}$$

in a different way.

We proceed to comment on the required identities for the simplicial homotopy. Recall the structure maps d_i and s_i after equation (1.2). We have $d_0 h_0(e) = \iota r(e)$ as follows. The top row of $h_0(e)$ is $* \rhd A_n = A_n = \cdots = A_n$, which becomes $* \rhd * \rhd \cdots \rhd *$ after modding out by A_n and contracting the initial $*$, as d_0 does. The middle row of $h_0(e)$ is $* \rhd X_0 \rhd X_1 \rhd \cdots \rhd X_n$, but since $X_0 = A_n$ and $X_k/A_n = B_k$ for $k \geq 1$, the second row of $h_0(e)$ becomes B after applying d_0 . The bottom row of $h_0(e)$ is B with an additional $*$ at the beginning, which is contracted by d_0 to return B .

At the other extreme, we have $d_{n+1} h_n(e) = \text{Id}_{f/(m,A')}(e)$. The $(n+1)$ -simplex $h_n(e)$ is (1.10) with an additional column on the right which reads $A_n \rhd X_n \rhd B_n$. The face map d_{n+1} truncates this additional column to give e .

The identity $d_i h_j(e) = h_{j-1} d_i(e)$ for $i = 0$ and $0 < j$ goes as follows. This identity is true for the top row, as modding out the top row of $h_j(e)$ by A_1 and contracting $*$ is the same as modding out the top row of e by A_1 , contracting $*$, and then replacing the entries starting at j by A_n/A_1 . For the middle row, we observe that the definition (1.12) with $s = 1$ is the identity $d_0 h_j(e) = h_{j-1} d_0(e)$ for $0 < j$. For the bottom row of the identity $d_0 h_j(e) = h_{j-1} d_0(e)$ with $0 < j$, we observe that modding out by B_1 and inserting $B_j/B_1 = B_j/B_1$ is the same as inserting $B_j = B_j$ and then modding out by B_1 .

The identity $d_i h_j(e) = h_{j-1} d_i(e)$ for $0 \neq i < j$ is not difficult to see.

For the identity $d_i h_j(e) = d_i h_{i-1}(e)$ for $i = j \neq 0$, we note that composing away the middle column in the two subdiagrams of $h_i(e)$

and $h_{i-1}(e)$ below, yields the same outcome.

$$\begin{array}{ccccc}
 A_{i-1} & \xrightarrow{\quad} & A_i & \xrightarrow{\quad} & A_n \\
 \downarrow & & \downarrow & & \downarrow \\
 C_{i-1} & \xrightarrow{\quad} & C_i & \xrightarrow{\quad} & X_i \\
 \downarrow & & \downarrow & & \downarrow \\
 B_{i-1} & \xrightarrow{\quad} & B_i & \xrightarrow{\quad} & B_i
 \end{array}
 \qquad
 \begin{array}{ccccc}
 A_{i-1} & \xrightarrow{\quad} & A_n & \xrightarrow{\quad} & A_n \\
 \downarrow & & \downarrow & & \downarrow \\
 C_{i-1} & \xrightarrow{\quad} & X_{i-1} & \xrightarrow{\quad} & X_i \\
 \downarrow & & \downarrow & & \downarrow \\
 B_{i-1} & \xrightarrow{\quad} & B_{i-1} & \xrightarrow{\quad} & B_i
 \end{array}$$

The remaining simplicial homotopy identities

$$\begin{aligned}
 d_i h_j &= h_j d_{i-1} \quad \text{for } i > j + 1 \\
 s_i h_j &= \begin{cases} h_{j+1} s_i & \text{for } i \leq j \\ h_j s_{i-1} & \text{for } i > j \end{cases}
 \end{aligned}$$

are not difficult to see. \square

Waldhausen proved in [32, Proposition 1.3.2 (1)] that the following Additivity Theorem is equivalent to the special case $\mathcal{A} = \mathcal{B} = \mathcal{C}$ in Theorem 1.3 above. On the other hand, we conclude the following Additivity Theorem directly from Lemma 1.9, which was already formulated for $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$.

Theorem 1.15 (Equivalent Formulation of Waldhausen Additivity). *Let \mathcal{C} be a Waldhausen category and \mathcal{A} and \mathcal{B} sub Waldhausen categories. Then the projection*

$$wS_{\bullet}(s, q): wS_{\bullet}\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \longrightarrow wS_{\bullet}\mathcal{A} \times wS_{\bullet}\mathcal{B}$$

is a weak equivalence of simplicial objects in \mathbf{Cat} . That is, the diagonal of the level-wise nerve is a weak equivalence of simplicial sets.

Proof. We conclude this from the weak equivalence of object simplicial sets in Lemma 1.9 in exactly the same way that Waldhausen, on page 336 of [32], concludes his Theorem 1.4.2 (recalled here in Theorem 1.3) from object simplicial sets in the case $\mathcal{A} = \mathcal{B} = \mathcal{C}$ of Lemma 1.9. See our proof of Theorem 4.11 for the quasicategorical version. \square

2. CLASSICAL ADDITIVITY AND SPLIT-EXACT SEQUENCES OF WALDHAUSEN CATEGORIES

We explain how Waldhausen Additivity, as recalled in Theorem 1.3, implies that the algebraic K -theory functor takes split-exact sequences to split cofiber sequences.

Definition 2.1 (Exact sequence of Waldhausen categories). Let \mathcal{A} , \mathcal{E} , and \mathcal{B} be Waldhausen categories. A sequence of exact functors

$$(2.2) \qquad \mathcal{A} \xrightarrow{i} \mathcal{E} \xrightarrow{f} \mathcal{B}$$

is called *exact* if

- (i) the composite $f \circ i$ is the distinguished zero object $*$ of \mathcal{B} ,
- (ii) the exact functor $i: \mathcal{A} \rightarrow \mathcal{E}$ is fully faithful, and
- (iii) the restricted functor $f|_{\mathcal{E}/\mathcal{A}}: \mathcal{E}/\mathcal{A} \rightarrow \mathcal{B}$ is an equivalence of categories. Here \mathcal{E}/\mathcal{A} is the full subcategory of \mathcal{E} on the objects $E \in \mathcal{E}$ such that $\mathcal{E}(i(A), E)$ is a point for all $A \in \mathcal{A}$.

Definition 2.3 (Split-exact sequence of Waldhausen categories). An exact sequence of Waldhausen categories and exact functors as in equation (2.2) is called *split* if there exist exact functors

$$\mathcal{A} \xleftarrow{j} \mathcal{E} \xleftarrow{g} \mathcal{B}$$

right adjoint to i and f respectively, such that the unit $\text{Id}_{\mathcal{A}} \rightarrow ji$ and the counit $fg \rightarrow \text{Id}_{\mathcal{B}}$ are natural isomorphisms.

Remark 2.4. In a split-exact sequence of Waldhausen categories and exact functors, the functor g is actually an inverse equivalence to $f|_{\mathcal{E}/\mathcal{A}}$. We have $\mathcal{E}(i(A), g(B)) \cong \mathcal{B}(f(i(A)), B) \cong \mathcal{B}(*, B) = \text{pt}$ for all A in \mathcal{A} , so that g goes into \mathcal{E}/\mathcal{A} . The counit $(f|_{\mathcal{E}/\mathcal{A}})g \rightarrow \text{Id}_{\mathcal{B}}$ is a natural isomorphism by hypothesis, and the unit $\text{Id}_{\mathcal{E}/\mathcal{A}} \rightarrow g(f|_{\mathcal{E}/\mathcal{A}})$ is a natural isomorphism since the left adjoint $f|_{\mathcal{E}/\mathcal{A}}$ is fully faithful. See also [12, Lemma 9.26].

Remark 2.5. In a split-exact sequence of Waldhausen categories and exact functors, the composite $j \circ g$ is naturally isomorphic to the distinguished zero object. The functor g goes into \mathcal{E}/\mathcal{A} by Remark 2.4, so for all $A \in \mathcal{A}$ we have $\mathrm{pt} = \mathcal{E}(i(A), g(B)) \cong \mathcal{A}(A, jg(B))$, and $jg(B)$ is a terminal object of \mathcal{A} , so isomorphic to $*$.

Example 2.6. Any Waldhausen category \mathcal{C} with selected sub Waldhausen categories \mathcal{A} and \mathcal{B} produces a split-exact sequence as follows. Let \mathcal{E} be the Waldhausen category $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ in Notation 1.4. We define exact functors

$$(2.7) \quad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) & \xrightarrow{q} & \mathcal{B} \\ & \swarrow s & & \nwarrow g & \\ & & & & \end{array}$$

by

$$\begin{aligned} i(A) &= \left(A \xrightarrow{=} A \twoheadrightarrow * \right) \\ s \left(A \twoheadrightarrow C \twoheadrightarrow B \right) &= A \\ q \left(A \twoheadrightarrow C \twoheadrightarrow B \right) &= B \\ g(B) &= \left(* \twoheadrightarrow B \xrightarrow{=} B \right). \end{aligned}$$

Clearly, $q \circ i = *$. The unit and counit for the adjunction $i \dashv s$ are $\eta_A = 1_A: A \rightarrow siA$ and

$$\varepsilon_{ABC}: is(ACB) \rightarrow ACB \quad \begin{array}{ccccc} A & \xrightarrow{=} & A & \twoheadrightarrow & * \\ \downarrow & & \downarrow m & & \downarrow \\ A & \xrightarrow{m} & C & \twoheadrightarrow & B \end{array}$$

Since $s\varepsilon_A = 1_A$ and $\varepsilon_{iA} = 1_{iA}$, the triangle identities clearly hold. Since the unit is a natural isomorphism, the left adjoint i is fully faithful.

The unit and the counit for the adjunction $q \dashv g$ are

$$\eta_{ABC}: ACB \rightarrow gq(ACB) \quad \begin{array}{ccccc} A & \longrightarrow & C & \xrightarrow{n} & B \\ \downarrow & & \downarrow n & & \downarrow = \\ * & \longrightarrow & B & \xrightarrow{=} & B \end{array}$$

and $\varepsilon_B = 1_B: qgB \rightarrow B$. Since $\eta_{gB} = 1_{gB}$ and $q\eta_{ACB} = 1_B$, the triangle identities clearly hold. Since the counit is a natural isomorphism, the right adjoint g is full faithful.

The subcategory \mathcal{E}/\mathcal{A} is full on the cofiber sequences isomorphic to those of the form

$$(2.8) \quad * \twoheadrightarrow B \xrightarrow{=} B$$

with $B \in \mathcal{B}$, since the condition that there be only one map

$$(2.9) \quad \begin{array}{ccccc} A & \xrightarrow{=} & A & \twoheadrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & C' & \twoheadrightarrow & B' \end{array}$$

for each $A \in \mathcal{A}$ implies that A' is a terminal object of \mathcal{A} (the middle map is determined by the left map), so A' is then isomorphic to the zero object of \mathcal{A} (hence also to that of \mathcal{C}), so that $C' \twoheadrightarrow B'$ is an isomorphism (it is a pushout of the isomorphism $A' \rightarrow *$). The isomorphism $C' \twoheadrightarrow B'$ is part of an isomorphism from the bottom row of (2.9) to $* \twoheadrightarrow B' \twoheadrightarrow B'$, which is of the form (2.8).

So $g: \mathcal{B} \rightarrow \mathcal{E}/\mathcal{A}$ is essentially surjective, and an equivalence. Thus its adjoint $f|_{\mathcal{E}/\mathcal{A}}: \mathcal{E}/\mathcal{A} \rightarrow \mathcal{B}$ is also an equivalence, and the sequence (2.7) is split-exact.

Proposition 2.10. *Suppose a split-exact sequence of Waldhausen categories and exact functors*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E} \xrightarrow{f} \mathcal{B} \\ \curvearrowleft & & \curvearrowleft \\ & j & g \end{array}$$

has the following three properties.

- (i) *Each counit component $ij(E) \rightarrow E$ is a cofibration.*

(ii) For each morphism $E \rightarrow E'$ in \mathcal{E} , the induced map

$$E \cup_{ij(E)} ij(E') \rightarrow E'$$

is a cofibration in \mathcal{E} .

(iii) In every cofiber sequence in \mathcal{A} of the form $A_0 \rightarrowtail A_1 \twoheadrightarrow *$, the first map is an isomorphism.

Then it is Waldhausen equivalent (Definition 6.1) to a split-exact sequence of the form (2.7) in Example 2.6.

Proof. Let $\mathcal{C} := \mathcal{E}$ and let $\mathcal{A}', \mathcal{B}' \subseteq \mathcal{C}$ be the essential images of \mathcal{A} and \mathcal{B} in \mathcal{C} under i and g respectively.

Since the unit $\text{Id}_{\mathcal{A}} \rightarrow ji$ and counit $fg \rightarrow \text{Id}_{\mathcal{B}}$ are natural isomorphisms, the left adjoint i and the right adjoint g are fully faithful, and therefore provide equivalences with the essential images \mathcal{A}' and \mathcal{B}' .

$$(2.11) \quad \begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{f} & \mathcal{B} \\ i \downarrow & & \Phi \downarrow & & \downarrow g \\ \mathcal{A}' & \longrightarrow & \mathcal{E}(\mathcal{A}', \mathcal{C}, \mathcal{B}') & \longrightarrow & \mathcal{B}' \end{array}$$

The vertical functors i and g are Waldhausen equivalences because they are exact equivalences which reflect weak equivalences and cofibrations (see Proposition 6.2). Reflectivity for weak equivalences and cofibrations follows from the unit $\text{Id}_{\mathcal{A}} \rightarrow ji$ and counit $fg \rightarrow \text{Id}_{\mathcal{B}}$ natural isomorphisms and the exactness of j and f .

Let $\Phi(E)$ be

$$(2.12) \quad ij(E) \rightarrowtail E \xrightarrow{\text{unit}} gf(E).$$

We claim that (2.12) is in fact a cofiber sequence. Consider the pushout P in the left diagram below.

$$(2.13) \quad \begin{array}{ccc} ij(E) \xrightarrow{\text{counit}} E & & \\ \downarrow \text{pushout} & \searrow \text{unit} & \\ * \xrightarrow{\quad} P & \xrightarrow{\quad} gf(E) & \end{array} \quad \begin{array}{ccc} j ij(E) \xrightarrow{j(\text{counit})} j(E) & & \\ \downarrow \text{pushout} & \searrow j(\text{unit}) & \\ * \xrightarrow{\quad} j(P) & \xrightarrow{\quad} jgf(E) & \end{array}$$

(Note: In the original image, dashed arrows labeled $\exists!$ connect P to $gf(E)$ and $j(P)$ to $jgf(E)$ in both diagrams.)

The outer square of the left diagram commutes because g goes in \mathcal{E}/\mathcal{A} by Remark 2.4, and $j(E) \in \mathcal{A}$. The right diagram is obtained by applying the exact functor j to the entire left diagram. By a triangle identity for the i, j adjunction, the map $j(\text{counit})$ in the right diagram

is an isomorphism, so that $* \rightharpoonup j(P)$ is also an isomorphism. For all $A \in \mathcal{A}$, $\mathcal{E}(i(A), P) \cong \mathcal{A}(A, j(P)) \cong \mathcal{A}(A, *) = \text{pt}$, so that $P \in \mathcal{E}/\mathcal{A}$. By Remark 2.4, there exists some $Q \in \mathcal{B}$ such that $P \cong g(Q)$.

Consider now the left diagram with P replaced by $g(Q)$, and its right vertical map and dashed map adjusted accordingly to have a pushout and the induced map. Apply f to the entire altered left diagram to obtain

$$(2.14) \quad \begin{array}{ccc} * & \xrightarrow{f(\text{counit})} & f(E) \\ \downarrow & \text{pushout} & \downarrow \\ * & \xrightarrow{\quad} & fg(Q) \\ & \searrow & \downarrow f(\text{unit}) \\ & & fgf(E) \end{array}$$

$\xrightarrow{\exists!}$

The vertical map $f(E) \rightarrow fg(Q)$ is an isomorphism, as it is a pushout of an isomorphism. By a triangle identity for the f, g adjunction, the map $f(\text{unit})$ is an isomorphism, so the dashed map $fg(Q) \dashrightarrow fgf(E)$ is also an isomorphism by 3-for-2. Its origin, the dashed map $g(Q) \dashrightarrow gf(E)$ in the altered left diagram, is in the image of the fully faithful functor g , and hence in \mathcal{E}/\mathcal{A} . Since $f|_{\mathcal{E}/\mathcal{A}}$ reflects isomorphisms (it is fully faithful), the map $g(Q) \dashrightarrow gf(E)$ is also an isomorphism. Finally, the outer square of the altered left diagram is isomorphic to a pushout square and (2.12) is a cofiber sequence.

The functor Φ defined as in (2.12) is exact by hypothesis (ii) and the exactness of ij and gf .

We define $\Psi: \mathcal{E}(\mathcal{A}', \mathcal{C}, \mathcal{B}') \rightarrow \mathcal{E}$ to be the projection to $\mathcal{C} = \mathcal{E}$. Clearly $\Psi \circ \Phi = \text{Id}_{\mathcal{E}}$, so that Φ is faithful. For fullness, a morphism $\Phi(E) \rightarrow \Phi(F)$ must come from the associated map $E \rightarrow F$ by the universality of $ij(F) \rightharpoonup F$ and the universality of $E \rightharpoonup gf(E)$, as well as the fully faithfulness of i and g .

To show $\Phi: \mathcal{E} \rightarrow \mathcal{E}(\mathcal{A}', \mathcal{C}, \mathcal{B}')$ is essentially surjective, it suffices to prove that the top row of the middle diagram

$$(2.15) \quad \begin{array}{ccccccc} A & & i(A) \rightharpoonup & E & \twoheadrightarrow & g(B) & B \\ \downarrow \exists! \downarrow m & & \downarrow i(m) & \parallel & & \uparrow \downarrow g(n) & \uparrow n \downarrow \exists! \\ j(E) & & ij(E) \rightharpoonup & E & \twoheadrightarrow & gf(E) & f(E) \end{array}$$

is isomorphic to the bottom row of the middle diagram via the dashed arrows coming from the universality of the counit and unit. An application of f to the middle diagram maps both left horizontal arrows to $* \rightharpoonup f(E)$ and therefore the right horizontal arrows to isomorphisms. Thus $f(g(n))$ is an isomorphism, and $g(n)$ is an isomorphism (f is an

equivalence of categories on \mathcal{E}/\mathcal{A} , so reflects isomorphisms there). On the other hand, an application of j to the middle diagram maps both right horizontal arrows to $j(E) \rightarrow *'$ by Remark 2.5, and therefore j maps the left horizontal arrows to isomorphisms by hypothesis. Therefore $ji(m)$ is an isomorphism, and m is also an isomorphism using the unit isomorphism of the i, j adjunction.

The exact equivalence Φ reflects weak equivalences and cofibrations (simply project to $\mathcal{C} = \mathcal{E}$), so Φ is a Waldhausen equivalence by Proposition 6.2. \square

Remark 2.16. Conditions (i) and (iii) in Proposition 2.10 hold if \mathcal{E} and \mathcal{A} are assumed to be *stable* categories, that is, cofiber sequences and fiber sequences coincide, or, more strongly, pushout squares and pullback squares coincide.

Remark 2.17. In the proof of Proposition 2.10, the diagrams

$$(2.18) \quad \begin{array}{ccccc} \mathcal{A} & \xleftarrow{j} & \mathcal{E} & \xrightarrow{f} & \mathcal{B} \\ \downarrow i & & \downarrow \Phi & & \downarrow g \\ \mathcal{A}' & \xleftarrow{s} & \mathcal{E}(\mathcal{A}', \mathcal{C}, \mathcal{B}') & \xrightarrow{q} & \mathcal{B}' \end{array}$$

commute strictly, while the diagrams

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E} & \xleftarrow{g} & \mathcal{B} \\ \downarrow i & & \downarrow \Phi & & \downarrow g \\ \mathcal{A}' & \longrightarrow & \mathcal{E}(\mathcal{A}', \mathcal{C}, \mathcal{B}') & \longleftarrow & \mathcal{B}' \end{array}$$

commute only up to natural isomorphism.

Theorem 2.19 (General Waldhausen Additivity). *Suppose a split-exact sequence of Waldhausen categories and exact functors*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{f} & \mathcal{B} \\ \curvearrowleft & & \curvearrowleft & & \\ & j & & g & \end{array}$$

has the following three properties.

- (i) Each counit component $ij(E) \rightarrow E$ is a cofibration.
- (ii) For each morphism $E \rightarrow E'$ in \mathcal{E} , the induced map

$$E \cup_{ij(E)} ij(E') \rightarrow E'$$

is a cofibration in \mathcal{E} .

- (iii) In every cofiber sequence in \mathcal{A} of the form $A_0 \rightarrowtail A_1 \rightarrow *$, the first map is an isomorphism.

Then the map

$$wS_{\bullet}(j, f): wS_{\bullet}\mathcal{E} \longrightarrow wS_{\bullet}\mathcal{A} \times wS_{\bullet}\mathcal{B}$$

is a weak equivalence of simplicial objects in \mathbf{Cat} . That is, the diagonal of the level-wise nerve is a weak equivalence of simplicial sets.

Proof. The vertical maps in the strictly commutative diagram (2.18) are Waldhausen equivalences by Proposition 2.10, so they induce weak homotopy equivalences after application of wS_\bullet by Corollary 6.7. The result now follows from Theorem 1.15, and the 3-for-2 property of weak equivalences between spaces. A finite product of weak equivalences of simplicial sets is also a weak equivalence because geometric realization and π_* preserve finite products. \square

Let $K(\mathcal{C})$ denote the K -theory spectrum of the Waldhausen category \mathcal{C} , that is $K(\mathcal{C})_n := |wS_\bullet \cdots S_\bullet \mathcal{C}|$ for $n \geq 0$ where S_\bullet appears n times. This is an Ω -spectrum beyond the 0-th term, that is, $|w\mathcal{C}| \rightarrow \Omega|wS_\bullet \mathcal{C}|$ might not be a weak equivalence of spaces, though $|wS_\bullet^{(n)} \mathcal{C}| \rightarrow \Omega|wS_\bullet^{(n+1)} \mathcal{C}|$ is a weak equivalence of spaces for $n \geq 1$.

Theorem 2.20 (General Waldhausen Additivity, Spectral Form). *Suppose a split-exact sequence of Waldhausen categories and exact functors*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{f} & \mathcal{B} \\ & \searrow j & \swarrow g & & \end{array}$$

has the following three properties.

- (i) Each counit component $ij(E) \rightarrow E$ is a cofibration.
- (ii) For each morphism $E \rightarrow E'$ in \mathcal{E} , the induced map

$$E \cup_{ij(E)} ij(E') \rightarrow E'$$

is a cofibration in \mathcal{E} .

- (iii) In every cofiber sequence in \mathcal{A} of the form $A_0 \twoheadrightarrow A_1 \twoheadrightarrow *$, the first map is an isomorphism.

Then the functors j and f induce a stable equivalence of K -theory spectra

$$K(j, f): K(\mathcal{E}) \longrightarrow K(\mathcal{A}) \vee K(\mathcal{B}).$$

Proof. In spectra, the wedge product and the product are formed levelwise. The inclusion of the wedge product of two spectra into the product is a stable equivalence. An application of S_\bullet to a split-exact sequence of Waldhausen categories and exact functors gives a simplicial split-exact sequence of Waldhausen categories and exact functors, so we may apply Theorem 2.19 to the individual levels of the K -theory spectra and obtain a level-wise (hence also stable) equivalence from $K(\mathcal{E})$ to $K(\mathcal{A}) \times K(\mathcal{B})$. The result then follows from the 3-for-2 property of stable equivalences. \square

Remark 2.21. The K -theory spectrum can actually be made into a symmetric spectrum, see the Appendix of Geisser–Hesselholt [14], or [4, 2.2], [3, A.5.4], [28, Section 2].

3. RECOLLECTIONS ABOUT QUASICATEGORIES

We summarize the few results about simplicial sets, quasicategories, Kan complexes, and groupoids that we will freely use from Joyal's papers [19] and [20] and Lurie's books [21] and [22]. We also prove some basic results about adjunctions and equivalences of quasicategories and simplicial categories. Pushouts are also discussed. To make the present paper as self-contained as possible, we have attempted to use only fundamental aspects of quasicategories and to prove everything else from these.

3.1. Quasicategories and Equivalences in Quasicategories. A *quasicategory* is a simplicial set X in which every inner horn admits a filler. That is, for any $0 < k < n$ and any map $\Lambda^k[n] \rightarrow X$, there exists a map $\Delta[n] \dashrightarrow X$ such that the diagram

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{\quad} & X \\ \downarrow & \nearrow \exists & \\ \Delta[n] & & \end{array}$$

commutes. The notion is originally due to Boardman and Vogt [5] under the name of *weak Kan complex*, and has been developed extensively by Joyal [20] and Lurie [21], [22]. Any Kan complex, for example the singular complex of a topological space, is a quasicategory. The nerve of any category is also a quasicategory. An *object of a quasicategory* X is a vertex of X , that is, an element of X_0 . A *map* or *morphism in* X is a 1-simplex f , its *source* is $d_1 f$ and its *target* is $d_0 f$. If X is a quasicategory and Y is a simplicial subset of X which is 0-full (see below) in X on its set of vertices Y_0 , then Y is also a quasicategory [20, page 275]. A *subquasicategory* of a quasicategory is a simplicial subset which is also a quasicategory.

If X is a quasicategory and A is a simplicial set, then the mapping space $X^A = \text{Map}(A, X)$ is a quasicategory [20, Corollary 2.19]. As usual, the n -simplices of the mapping space are

$$(X^A)_n = \text{Map}(A, X)_n = \mathbf{S}\mathbf{Set}(A \times \Delta[n], X)$$

and the mapping space construction is right adjoint to the product, that is there is bijection

$$(3.1) \quad \mathbf{S}\mathbf{Set}(A \times B, C) \cong \mathbf{S}\mathbf{Set}(A, C^B)$$

natural in simplicial sets A , B , and C . If X is a quasicategory and A is a simplicial set, we write $\text{Fun}(A, X)$ for X^A and call $\text{Fun}(A, X)$ the *quasicategory of functors from* A *to* X . The 0-simplices of $\text{Fun}(A, X)$ are the *functors* from A to X , these are simply maps of simplicial sets $A \rightarrow X$. The 1-simplices of $\text{Fun}(A, X)$ are called *natural transformations*. These are the maps of simplicial sets $A \times \Delta[1] \rightarrow X$.

For any two objects x and y of a quasicategory X , we have the *mapping space* $X(x, y)$ of the quasicategory X . This simplicial set is defined as the following pullback.

$$(3.2) \quad \begin{array}{ccc} X(x, y) & \longrightarrow & X^{\Delta[1]} \\ \downarrow & \text{pullback} & \downarrow (s, t) \\ * & \xrightarrow{(x, y)} & X \times X \end{array}$$

We use Joyal's notation $X(x, y)$, see [20, page 158]. Instead of $X(x, y)$, Lurie writes $\mathrm{Hom}_S(x, y)$, see [21, page 28]. If \mathcal{C} is a category, the mapping space $(N\mathcal{C})(x, y)$ is the same as $\mathcal{C}(x, y)$ viewed as a discrete simplicial set. The mapping spaces $X(x, y)$ are Kan complexes [6, Proposition 6.13], which are ∞ -groupoids, so in this sense quasicategories are models for $(\infty, 1)$ -categories. Two vertices of $X(x, y)$ (morphisms of X) are *homotopic* if there is a 1-simplex of $X(x, y)$ from one to the other; this is an equivalence relation on $X(x, y)$. The path components of $X(x, y)$ are the homotopy classes of 1-morphisms in X from x to y . Since X is a quasicategory, any two morphisms in X are homotopic in the above sense if and only if they are *left homotopic*, which is the case if and only if they are *right homotopic*, see [20, Definition 1.8 and Lemma 1.9] and [5].

One drawback of this choice of mapping space $X(x, y)$ as the pullback in (3.2) is that there is no composition

$$(3.3) \quad X(y, z) \times X(x, y) \longrightarrow X(x, z) .$$

For this reason, we discuss *simplicial categories* later. For a treatment of mapping spaces, see Dugger–Spivak [9].

A map of simplicial sets $f: A \rightarrow B$ is *n-full* if the unit naturality square

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{cosk}_n \mathrm{tr}_n A \\ f \downarrow & & \downarrow \mathrm{cosk}_n \mathrm{tr}_n f \\ B & \longrightarrow & \mathrm{cosk}_n \mathrm{tr}_n B \end{array}$$

is a pullback [20, Definition B.0.11]. Here cosk_n is the right adjoint to the n -th truncation functor tr_n . A simplicial subset of a simplicial set is *n-full* if the inclusion functor is *n-full*. The map f is *n-full* if and only if it has the right lifting property with respect to the inclusion $\partial\Delta[m] \hookrightarrow \Delta[m]$ for all $m > n$ [20, Proposition B.0.12]. Thus, *n-full* implies $(n+1)$ -full.

The left adjoint τ_1 to the fully faithful nerve functor $N: \mathbf{Cat} \rightarrow \mathbf{SSet}$ sends a simplicial set A to the quotient of the free category on the graph (A_0, A_1, d_0, d_1) by the relations determined by 2-simplices [13]. If X is a quasicategory, then $\tau_1 X$ coincides with the Boardman–Vogt homotopy category hoX and $\pi_0 X(x, y) = (\tau_1 X)(x, y)$ [20, Proposition

1.11, page 213]. A useful fact about the functor τ_1 is that it preserves finite products [20, B.0.15].

A morphism f in a quasicategory X is said to be an *equivalence* if its image in the homotopy category $\tau_1 X$ is invertible.³ The equivalences in a quasicategory X satisfy the 3-for-2 property. That is, if $x: \Delta[2] \rightarrow X$ is a 2-simplex in X such that any two of the morphisms $\sigma_{0,1}^*(x)$, $\sigma_{1,2}^*(x)$, $\sigma_{0,2}^*(x)$ is an equivalence, then so is the third. This follows from the 3-for-2 property of isomorphisms in $\tau_1 X$ and the fact that a 2-simplex in X gives rise to a commutative triangle in $\tau_1 X$.

A quasicategory X is a Kan complex if and only if $\tau_1 X$ is a groupoid, see Joyal [19, Corollary 1.4] and [20, Theorem 4.14].

The inclusion of small groupoids into small categories, $\mathbf{Grpd} \rightarrow \mathbf{Cat}$, admits a right adjoint $(-)_{\text{iso}}$. It assigns to a category \mathcal{C} the maximal groupoid contained in \mathcal{C} . The maximal groupoid \mathcal{C}_{iso} has the same objects as \mathcal{C} , but the morphisms are all isomorphisms in \mathcal{C} . Similarly, the inclusion of small Kan complexes into small quasicategories, $\mathbf{Kan} \rightarrow \mathbf{QCat}$, admits a right adjoint denoted $(-)_{\text{equiv}}$.⁴ For a quasicategory X , the simplicial set X_{equiv} is the maximal Kan complex contained in X [20, Theorem 4.19].

The Kan complex X_{equiv} is 1-full in the quasicategory X . An n -simplex $x: \Delta[n] \rightarrow X$ is in X_{equiv} if and only if the morphism $\sigma_{i,j}^*(x)$ is an equivalence in X for all $0 \leq i < j \leq n$, where $\sigma_{i,j}: [1] \rightarrow [n]$ is the relevant injection, [19, Corollary 1.5] and [20, Lemma 4.18]. In particular, the 1-simplices of X_{equiv} are the equivalences in X . Thus, X_{equiv} is the subcomplex of X 1-full on the morphisms in X which are invertible in the homotopy category hoX .

Lemma 3.4. *If \mathcal{C} is a category, then $(N\mathcal{C})_{\text{equiv}} = N(\mathcal{C}_{\text{iso}})$.*

Proof. Since $(N\mathcal{C})_{\text{equiv}}$ is the maximal Kan subcomplex of $N\mathcal{C}$, it contains the Kan subcomplex $N(\mathcal{C}_{\text{iso}})$.

A 1-simplex in $N\mathcal{C}$ is an equivalence in $N\mathcal{C}$ if and only if its image is an isomorphism in $\tau_1 N\mathcal{C} \cong \mathcal{C}$, so the 1-simplices of $(N\mathcal{C})_{\text{equiv}}$ and $N(\mathcal{C}_{\text{iso}})$ coincide. If x is an n -simplex of $(N\mathcal{C})_{\text{equiv}}$, then it is a path of n morphisms in \mathcal{C} in which the composites $\sigma_{i,j}^*(x)$ are equivalences in $N\mathcal{C}$ and isomorphisms in \mathcal{C} for all $0 \leq i < j \leq n$. In particular, each of the n morphisms is invertible in \mathcal{C} and x is in $N(\mathcal{C}_{\text{iso}})$, so that $(N\mathcal{C})_{\text{equiv}} \subseteq N(\mathcal{C}_{\text{iso}})$. \square

If A is a simplicial set such that $\tau_1 A$ is a groupoid, then every map $A \rightarrow X$ factors through the inclusion $X_{\text{equiv}} \subseteq X$ [20, Proposition 4.21].

The maximal Kan subcomplex X_{equiv} of the quasicategory X is more closely related to groupoids in the following straightforward extension

³Joyal prefers to call such a morphism an *isomorphism* in [20]: it is the quasi-categorical version of the categorical notion of isomorphism.

⁴Blumberg–Gepner–Tabuada write $(-)_{\text{iso}}$ instead of $(-)_{\text{equiv}}$ in [2].

of [20, Proposition 4.22]. Let $J[n]$ denote the nerve of the groupoid with objects $0, 1, \dots, n$ and a unique isomorphism from any object to another.

Proposition 3.5. *Let X be a quasicategory. An n -simplex $x: \Delta[n] \rightarrow X$ is in X_{equiv} if and only if it can be extended to a map $J[n] \rightarrow X$.*

Proof. Suppose x is in X_{equiv} . Then we have the following diagram, in which the right vertical arrow is a fibration.

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{x} & X_{\text{equiv}} \\ \downarrow & & \downarrow \\ J[n] & \longrightarrow & * \end{array}$$

The inclusion $\Delta[n] \hookrightarrow J[n]$ is a cofibration by definition, and a weak equivalence (both $\Delta[n]$ and $J[n]$ are contractible, as they are nerves of categories with a terminal object). Therefore the lifting problem $\bar{x}: J[n] \rightarrow X_{\text{equiv}}$ can be solved.

Suppose now x is in X and an extension $\bar{x}: J[n] \rightarrow X$ exists. The category $\tau_1(J[n])$ is isomorphic to a groupoid via the counit of the adjunction $\tau_1 \dashv N$, so $\bar{x}: J[n] \rightarrow X$ factors through the inclusion $X_{\text{equiv}} \subseteq X$. \square

Corollary 3.6. *If X is a quasicategory, then $(X_{\text{equiv}})_n = \mathbf{SSet}(J[n], X)$.*

Corollary 3.7. *Let X and Y be quasicategories and $\alpha: X \times \Delta[1] \rightarrow Y$ a natural transformation. Then the following are equivalent.*

- (i) α is a natural equivalence.
- (ii) α is invertible in the homotopy category $\tau_1 \text{Fun}(X, Y) = \tau_1(Y^X)$.
- (iii) α extends to a map $X \times J[1] \rightarrow Y$.
- (iv) Each component $\alpha_x := \alpha(s_0(x), 0 \rightarrow 1)$ is an equivalence in Y .

Proof. By definition, α is a natural equivalence if and only if it is an equivalence in $\text{Fun}(X, Y)$, which is the case if and only if $\alpha: \Delta[1] \rightarrow \text{Fun}(X, Y)$ extends to $\bar{\alpha}: J[1] \rightarrow \text{Fun}(X, Y)$. Thus, (i), (ii), and (iii) are equivalent.

By the functoriality and naturality of (3.1), α extends to $\bar{\alpha}$ if and only if the corresponding map $\alpha^X: X \rightarrow \text{Fun}(\Delta[1], Y)$ admits a lift to $\bar{\alpha}^X: X \rightarrow \text{Fun}(J[1], Y)$ along the restriction map $i^*: \text{Fun}(J[1], Y) \rightarrow \text{Fun}(\Delta[1], Y)$. Since the restriction i^* is a 0-full inclusion (it is the inclusion of the quasicategory of equivalences in Y to the quasicategory of morphisms in Y), the existence of a lift along i^* is the same as the existence of a lift on the level of 0-simplices, which is the same as requiring each component α_x to be an equivalence in Y . For a different proof of (i) \Leftrightarrow (iv), see [20, Theorem 5.14]. \square

3.2. Simplicial Categories and Homotopy Adjunctions. The mapping spaces $X(x, y)$ of a quasicategory X defined as in equation (3.2) do not have a composition as in equation (3.3). This is a disadvantage if one would like to develop characterizations of adjunctions in terms of universal properties of units and counits. To get around this, we now recall simplicial categories and develop a theory of weak adjunctions for them.

A *simplicial category* \mathcal{C} is a category enriched in simplicial sets. In a simplicial category, everything is strict. We thus have for each $a, b, c \in \text{ob } \mathcal{C}$, a map of simplicial sets

$$\mathcal{C}(b, c) \times \mathcal{C}(a, b) \xrightarrow{\circ} \mathcal{C}(a, c) .$$

All of these maps together make a strictly associative and strictly unital composition on \mathcal{C} . If $j \in \mathcal{C}(b, c)_0$, then “composition with j ” induces two maps of simplicial sets j_* , j^* , namely

$$\begin{array}{c} \mathcal{C}(a, b) \xrightarrow{\cong} \Delta[0] \times \mathcal{C}(a, b) \xrightarrow{j \times 1_{\mathcal{C}(a, b)}} \mathcal{C}(b, c) \times \mathcal{C}(a, b) \xrightarrow{\circ} \mathcal{C}(a, c) \\ \mathcal{C}(c, d) \xrightarrow{\cong} \mathcal{C}(c, d) \times \Delta[0] \xrightarrow{1_{\mathcal{C}(c, d)} \times j} \mathcal{C}(c, d) \times \mathcal{C}(b, c) \xrightarrow{\circ} \mathcal{C}(b, d) . \end{array}$$

j_* (top arc) and j^* (bottom arc)

If \mathcal{C} is a simplicial category, its *homotopy category* is $\pi_0 \mathcal{C}$, which is the category with the same objects as \mathcal{C} but with hom-sets $(\pi_0 \mathcal{C})(a, b)$ equal to the path components $\pi_0(\mathcal{C}(a, b))$. Two morphisms in \mathcal{C} (vertices of $\mathcal{C}(a, b)$) are *homotopic* if they are in the same path component of $\mathcal{C}(a, b)$. A morphism j in \mathcal{C} is an *equivalence* if it is invertible in the homotopy category $\pi_0 \mathcal{C}$. This is the case if and only if there exists a morphism j' in \mathcal{C} such that $j' \circ j$ and $j \circ j'$ are homotopic to the respective identity morphisms.

Lemma 3.8. *If $j, k \in \mathcal{C}(b, c)_0$ are homotopic as morphisms in \mathcal{C} , then j_* and k_* are homotopic as maps of simplicial sets, as are j^* and k^* .*

Proof. If $\alpha: \Delta[1] \rightarrow \mathcal{C}(b, c)$ is such that $\alpha_0 = j$ and $\alpha_1 = k$, then

$$\Delta[1] \times \mathcal{C}(a, b) \xrightarrow{\alpha \times 1_{\mathcal{C}(a, b)}} \mathcal{C}(b, c) \times \mathcal{C}(a, b) \xrightarrow{\circ} \mathcal{C}(a, c)$$

is a homotopy from j_* to k_* . But if j and k are merely in the same path component of $\mathcal{C}(b, c)$, then by definition they can be connected by a zig-zag of maps $\Delta[1] \rightarrow \mathcal{C}(b, c)$, which induce a zig-zag of homotopies, so that j_* and k_* are homotopic. \square

Proposition 3.9. *A morphism $j: b \rightarrow c$ in a simplicial category \mathcal{C} is an equivalence in \mathcal{C} if and only if $j_*: \mathcal{C}(a, b) \rightarrow \mathcal{C}(a, c)$ is a weak homotopy equivalence for all objects a in \mathcal{C} , which is the case if and*

only if $j^*: \mathcal{C}(c, d) \rightarrow \mathcal{C}(b, d)$ is a weak homotopy equivalence for all objects d in \mathcal{C} .

Proof. If j is an equivalence in \mathcal{C} with homotopy inverse j' , then j'_* is a homotopy inverse for j_* for all a by Lemma 3.8.

Suppose now that $j_*: \mathcal{C}(a, b) \rightarrow \mathcal{C}(a, c)$ is a weak homotopy equivalence for all objects a in \mathcal{C} . Then $\pi_0(j_*): \pi_0\mathcal{C}(a, b) \rightarrow \pi_0\mathcal{C}(a, c)$ is a bijection for all objects a in \mathcal{C} , so $\pi_0(j_*)$ is a natural isomorphism between the functors $\pi_0\mathcal{C}(-, b), \pi_0\mathcal{C}(-, c): (\pi_0\mathcal{C})^{\text{op}} \rightarrow \mathbf{Set}$. By the Yoneda Lemma, the morphism in $\pi_0\mathcal{C}$ which induces $\pi_0(j_*)$ via the Yoneda embedding for $\pi_0\mathcal{C}$ is then an isomorphism in $\pi_0\mathcal{C}$. But this morphism in $\pi_0\mathcal{C}$ is precisely the image of j in the homotopy category of \mathcal{C} . Hence, j is an equivalence in \mathcal{C} .

The proof of the statement for j^* is completely analogous. \square

See [21, Proposition 1.2.4.1, pages 33-34] for the topological version of Proposition 3.9 and further statements.

A simplicial functor $f: \mathcal{C} \rightarrow \mathcal{D}$ is *fully faithful* if each map of simplicial sets $f_{a,b}: \mathcal{C}(a, b) \rightarrow \mathcal{D}(fa, fb)$ is a weak homotopy equivalence. It is called *essentially surjective* if $\pi_0 f: \pi_0\mathcal{C} \rightarrow \pi_0\mathcal{D}$ is essentially surjective. We will never use the strict versions, so the adjectives fully faithful and essentially surjective as applied to simplicial functors are always meant in this homotopical sense. A fully faithful, essentially surjective simplicial functor is called a *Dwyer-Kan equivalence*.

Simplicial categories, simplicial functors, and (weak) natural transformations form a sesquicategory⁵ **SimpCat**. A (weak) natural transformation $\alpha: f \Rightarrow g$ of simplicial functors $\mathcal{C} \rightarrow \mathcal{D}$ assigns to each object x a morphism $\alpha_x: fx \rightarrow gx$ which is π_* -natural, that is, the two composite maps of simplicial sets

$$(3.10) \quad \mathcal{C}(x, y) \xrightarrow{f_{x,y}} \mathcal{D}(fx, fy) \xrightarrow{(\alpha_y)_*} \mathcal{D}(fx, gy)$$

$$(3.11) \quad \mathcal{C}(x, y) \xrightarrow{g_{x,y}} \mathcal{D}(gx, gy) \xrightarrow{(\alpha_x)^*} \mathcal{D}(fx, gy)$$

induce the same map of homotopy groups $\pi_n\mathcal{C}(x, y) \rightarrow \pi_n\mathcal{D}(fx, gy)$ for all $n \geq 0$ and all basepoints. Note that this is weaker than a simplicial functor $\alpha: \mathcal{C} \times [1] \rightarrow \mathcal{D}$ which restricts to f and g at 0 and 1. We will always use this weak notion of natural transformation between simplicial functors, so we will not write the adjective “weak”. A natural transformation is a *natural equivalence* if each of its components is an equivalence.

⁵A sesquicategory is like a 2-category, except there is not a chosen horizontal composition of 2-cells and the interchange law does not hold. Instead, there are two “whiskering” operations between morphisms and 2-cells, from which two horizontal compositions can be built. See [31] and [30].

An adjunction between simplicial categories \mathcal{C} and \mathcal{D} consists of simplicial functors $f: \mathcal{C} \rightarrow \mathcal{D}$ and $g: \mathcal{D} \rightarrow \mathcal{C}$ together with natural transformations $\eta: 1_{\mathcal{C}} \Rightarrow gf$ and $\varepsilon: fg \Rightarrow 1_{\mathcal{D}}$ such that the components of the triangle identities

$$(g * \varepsilon) \odot (\eta * g) = \text{Id}_g \quad (\varepsilon * f) \odot (f * \eta) = \text{Id}_f$$

are true in $\pi_0\mathcal{C}$ and $\pi_0\mathcal{D}$ respectively.

Proposition 3.12. *Simplicial functors $f: \mathcal{C} \rightarrow \mathcal{D}$ and $g: \mathcal{D} \rightarrow \mathcal{C}$ are adjoint if and only if simplicial sets $\varphi_{c,d}: \mathcal{D}(fc, d) \rightleftarrows \mathcal{C}(c, gd) : \psi_{c,d}$, natural in both variables, such that $\varphi_{c,d}\psi_{c,d}$ and $\psi_{c,d}\varphi_{c,d}$ induce identity maps in all homotopy groups at all basepoints for all objects c and d in \mathcal{C} and \mathcal{D} .⁶*

Proof. Suppose f and g are adjoint with unit η and counit ε . Define $\varphi(j) = gj \circ \eta_c$ and $\psi(k) = \varepsilon_d \circ fk$. Then $\varphi\psi(k) = g\varepsilon_d \circ gfk \circ \eta_c$ because g is a simplicial functor. But by naturality of η , this induces the same map in all homotopy groups at all base points as $k \mapsto g\varepsilon_d \circ \eta_{gd} \circ k$. By a triangle identity, the morphism $g\varepsilon_d \circ \eta_{gd}$ in \mathcal{C} is homotopic to the identity morphism on gd , so $(g\varepsilon_d \circ \eta_{gd})_*$ is homotopic to an identity map of simplicial sets by Lemma 3.8. Finally, $\varphi\psi$ induces the identity in homotopy groups. Similarly, $\psi\varphi$ induces the identity in homotopy groups.

For the converse, suppose f and g are functors between quasicategories and admit such φ and ψ . Since φ and ψ are natural, we may define natural transformations $\eta_c := \varphi(1_{fc})$ and $\varepsilon_d := \psi(1_{gd})$. By naturality of φ , the map $j \mapsto \varphi(j \circ 1_{fc})$ induces the same maps of homotopy groups as $j \mapsto gj \circ \eta_c$. Similarly, ψ induces the same maps of homotopy groups as $k \mapsto \varepsilon_d \circ fk$. Composing, we see that $\varphi \circ \psi$ induces the same map in homotopy groups as

$$k \mapsto g(\varepsilon_d \circ fk) \circ \eta_c = g\varepsilon_d \circ gfk \circ \eta_c,$$

which is the same as $k \mapsto g\varepsilon_d \circ \eta_{gd} \circ k$ in homotopy groups by naturality of η . Thus $(g\varepsilon_d \circ \eta_{gd})_*$ induces the identity map in all homotopy groups, in particular for π_0 . Then by the Yoneda Lemma for the category $\pi_0\mathcal{C}$, the morphisms $g\varepsilon_d \circ \eta_{gd}$ and 1_{gd} are the same in $\pi_0\mathcal{C}$. Consequently, the triangle identity $(g * \varepsilon) \odot (\eta * g) = \text{Id}_g$ holds in $\pi_0\mathcal{C}$. The other triangle identity can be similarly proved.

Indeed, these are just the appropriate simplicial variations on pages 80-84 of Mac Lane [23]. \square

Recall that a class of maps has the *6-for-2 property* if whenever we have any three composable maps $\xrightarrow{u} \xrightarrow{v} \xrightarrow{w}$ with vu and wv in the class,

⁶This is not equivalent to requiring a natural isomorphism in \mathbf{HoSSet} between $\mathcal{D}(fc, d)$ and $\mathcal{C}(c, gd)$, for that would only imply a zig-zag between $\mathcal{D}(fc, d)$ and $\mathcal{C}(c, gd)$ in \mathbf{SSet} . Moreover, a map of simplicial sets may induce identity in homotopy groups and still not be homotopic to the identity.

we can conclude that u , v , w , and wvu are also in the class. The class of bijections clearly has the 6-for-2 property. If φ and ψ are as in Proposition 3.12, and we consider $\pi_n(\varphi\psi\varphi)$, then we see φ and ψ are both weak homotopy equivalences.

An adjunction between simplicial categories induces an ordinary adjunction between their homotopy categories.

Proposition 3.13. *Suppose $f: \mathcal{C} \rightarrow \mathcal{D}$ and $g: \mathcal{D} \rightarrow \mathcal{C}$ are adjoint simplicial functors with unit $\eta: 1_{\mathcal{C}} \Rightarrow gf$ and counit $\varepsilon: fg \Rightarrow 1_{\mathcal{D}}$. Then the right adjoint g is fully faithful if and only if every component of the counit ε is an equivalence. Dually, the left adjoint f is fully faithful if and only if every component of the unit η is an equivalence.*

Proof. By naturality of ε , for any object e of \mathcal{D} the triangle

$$\begin{array}{ccc} \mathcal{D}(d, e) & \xrightarrow{g_{d,e}} & \mathcal{C}(gd, ge) \\ & \searrow (\varepsilon_d)^* \quad \swarrow \varepsilon_e f_{gd, ge}(-) & \\ & \mathcal{D}(fgd, e) & \end{array}$$

commutes in homotopy groups. The right morphism $\psi_{gd, e}(-) = \varepsilon_e \circ f_{gd, ge}(-)$ is a weak homotopy equivalence because f and g are adjoint (see the discussion after Proposition 3.12). By the 3-for-2 property of weak homotopy equivalences, $g_{d,e}$ is a weak homotopy equivalence if and only if $(\varepsilon_d)^*$ is a weak homotopy equivalence. The claim now follows from Proposition 3.9. \square

A simplicial functor $f: \mathcal{C} \rightarrow \mathcal{D}$ is called an *equivalence of simplicial categories* if there exists a simplicial functor $g: \mathcal{D} \rightarrow \mathcal{C}$ and natural equivalences $\eta: 1_{\mathcal{C}} \Rightarrow gf$ and $\varepsilon: fg \Rightarrow 1_{\mathcal{D}}$.

Proposition 3.14. *Any equivalence of simplicial categories is fully faithful and essentially surjective, i.e., a Dwyer-Kan equivalence.*

Proof. Let $f: \mathcal{C} \rightarrow \mathcal{D}$ be an equivalence of simplicial categories. Then the following two squares commute after application of π_n for all $n \geq 0$.

$$\begin{array}{ccccccc} & & & \text{Id}_{\mathcal{C}(b,c)} & & & \\ & & & \curvearrowright & & & \\ \mathcal{C}(b, c) & \xrightarrow{f_{b,c}} & \mathcal{D}(fb, fc) & \xrightarrow{g_{fb, fc}} & \mathcal{C}(gfb, gfc) & \xrightarrow[\cong]{(\eta_c^{-1})^*(\eta_b)^*} & \mathcal{C}(b, c) \\ & & \searrow & \downarrow f_{gfb, gfc} & & & \\ & & & \mathcal{D}(fgfb, fgfc) & & & \\ & & & \downarrow (\varepsilon_{fc})^*(\varepsilon_{fb}^{-1})^* \simeq & & & \\ & & & \mathcal{C}(fb, fc) & & & \\ & \searrow \text{Id}_{\mathcal{D}(fb, fc)} & & & & & \end{array}$$

We use the 6-for-2 property of bijections for π_n of the middle three arrows $f_{b,c}$, $g_{fb, fc}$, $f_{gfb, gfc}$ as follows. By the 3-for-2 property of bijections

and the commutativity of the squares involving the curved arrows, we see that $\pi_n(g_{fb,fc} \circ f_{b,c})$ and $\pi_n(f_{gfb,gfc} \circ g_{fb,fc})$ are bijections. Then by 6-for-2 for bijections, $\pi_n(f_{b,c})$ is a bijection for all $n \geq 0$ and all $a, b \in \mathcal{C}$, and f is fully faithful.

For the essential surjectivity of f , we note that $\pi_0: \mathbf{SimpCat} \rightarrow \mathbf{Cat}$ takes an equivalence of simplicial categories to an equivalence of categories, as it is a 2-functor. So $\pi_0(f)$ is essentially surjective. \square

Simplicial sets, simplicial categories, and categories are connected by the adjoint functors below, as used in [9, 2.16] and in other papers on this topic.

$$\begin{array}{ccccc}
 & & \tau_1 & & \\
 & \swarrow & & \searrow & \\
 \mathbf{SSet} & \xleftrightarrow{\mathfrak{C}} & \mathbf{SimpCat} & \xleftrightarrow[\text{disc}]{\pi_0} & \mathbf{Cat} \\
 & \nwarrow_{N^{\text{simp}}} & & \nearrow & \\
 & & N & &
 \end{array}$$

Here N is the nerve functor of Grothendieck, the *categorification* τ_1 is its left adjoint, N^{simp} is the *homotopy coherent nerve* of Cordier-Porter [7], [8], and \mathfrak{C} is its left adjoint [21, Section 1.1.5], sometimes called *rigidification*. For any simplicial set X , the objects of $\mathfrak{C}(X)$ are the vertices of X . The functor π_0 forms the homotopy category $\pi_0\mathcal{C}$ of a simplicially enriched category \mathcal{C} , which has $(\pi_0\mathcal{C})(a, b) = \pi_0(\mathcal{C}(a, b))$ as above. The functor disc means to consider a category as a simplicial category with discrete hom simplicial sets.

The equalities $N = N^{\text{simp}}\text{disc}$ and $\tau_1 = \pi_0\mathfrak{C}$ hold. Consequently, parallel morphisms j and k in a quasicategory X are homotopic in X if and only if they are homotopic in the simplicial category $\mathfrak{C}(X)$. Furthermore, j is an equivalence in X if and only if j is an equivalence in $\mathfrak{C}(X)$.

A useful fact about the functor \mathfrak{C} is that it weakly preserves finite products, that is, the map of simplicial categories $\mathfrak{C}(X \times Y) \rightarrow \mathfrak{C}(X) \times \mathfrak{C}(Y)$ is a Dwyer-Kan equivalence [10, Proposition 6.2], so that the composite functor $\pi_n\mathfrak{C}$ actually preserves products. Consequently, a natural transformation of simplicial sets $\alpha: X \times \Delta[1] \rightarrow Y$ induces a natural transformation of simplicial functors $\mathfrak{C}(X) \rightarrow \mathfrak{C}(Y)$. Namely, the images of the components $\alpha_x = \alpha(s_0(x), 0 \rightarrow 1)$ in $\mathfrak{C}(Y)$ are π_* -natural because π_n of equations (3.10) and (3.11) for $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$ are then the naturality diagram for the (ordinary) natural transformation

$$\pi_n\mathfrak{C}(X) \times [1] \cong \pi_n\mathfrak{C}(X \times \Delta[1]) \xrightarrow{\pi_n\mathfrak{C}(\alpha)} \pi_n\mathfrak{C}(Y) .$$

Here $\pi_n\mathfrak{C}(X)$ means the category with object set $\text{ob } \mathfrak{C}(X)$ and hom sets $(\pi_n\mathfrak{C}(X))(x, y) = \pi_n(\mathfrak{C}(X)(x, y))$.

We have recalled two notions of mapping space associated to objects x and y of a quasicategory X : the pullback simplicial set $X(x, y)$ in equation (3.2) and the hom simplicial set $\mathfrak{C}(X)(x, y)$ from above. Fortunately these are the same in the following weak sense.

Theorem 3.15 (Corollary 5.3 of [9]). *If X is a quasicategory and x, y are objects of X , then there is a natural zig-zag of weak homotopy equivalences between $X(x, y)$ and $(\mathfrak{C}X)(x, y)$.*

3.3. Adjunctions and Equivalences between Quasicategories.

We also need the notions of adjoint functors and equivalences between quasicategories. These are most easily phrased in terms of Joyal's 2-category of simplicial sets as in [20], denoted by \mathbf{SSet}^{τ_1} . Objects are simplicial sets, and morphisms are maps of simplicial sets. The 2-cells are given by the definition of hom categories $\mathbf{SSet}^{\tau_1}(A, B) := \tau_1(B^A)$. If X and Y are quasicategories, then a 2-cell between functors $j, k: X \rightarrow Y$ is the homotopy class of a functor $\alpha: X \times \Delta[1] \rightarrow Y$ with $\alpha_0 = j$ and $\alpha_1 = k$.

Proposition 3.16 (Proposition 1.27 of [20]). *The functor*

$$\tau_1: \mathbf{SSet}^{\tau_1} \longrightarrow \mathbf{Cat}$$

is a 2-functor. Hence it takes an equivalence in \mathbf{SSet}^{τ_1} to an equivalence of categories, an adjunction to an adjunction, and a left respectively right adjoint to a left respectively right adjoint.

An adjunction between quasicategories \mathcal{C} and \mathcal{D} is an adjunction between \mathcal{C} and \mathcal{D} in the 2-category \mathbf{SSet}^{τ_1} . More explicitly, an adjunction consists of functors $f: \mathcal{C} \rightarrow \mathcal{D}$ and $g: \mathcal{D} \rightarrow \mathcal{C}$, functors $\eta: \mathcal{C} \times \Delta[1] \rightarrow \mathcal{C}$ and $\varepsilon: \mathcal{D} \times \Delta[1] \rightarrow \mathcal{D}$ with $\eta_0 = \text{Id}_{\mathcal{C}}$, $\eta_1 = g \circ f$, $\varepsilon_0 = f \circ g$, and $\varepsilon_1 = \text{Id}_{\mathcal{D}}$, such that the triangle identities hold in the 2-category \mathbf{SSet}^{τ_1} :

$$(3.17) \quad (g * [\varepsilon]) \odot ([\eta] * g) = \text{Id}_g \quad ([\varepsilon] * f) \odot (f * [\eta]) = \text{Id}_f.$$

Here $[\eta]$ and $[\varepsilon]$ indicate the morphisms in the categories $\tau_1(\mathcal{C}^{\mathcal{C}})$ and $\tau_1(\mathcal{D}^{\mathcal{D}})$ associated to the morphism η and ε in $\text{Fun}(\mathcal{C}, \mathcal{C})$ and $\text{Fun}(\mathcal{D}, \mathcal{D})$. An adjunction between quasicategories induces an ordinary adjunction between their homotopy categories.

Since the equality $\tau_1 = \pi_0 \mathfrak{C}$ holds, the triangle identities (3.17) hold in \mathbf{SSet}^{τ_1} if and only if their \mathfrak{C} -images hold. Thus, functors and natural transformations f, g, η, ε between quasicategories form an adjunction if and only if the simplicial functors and natural transformations $\mathfrak{C}f, \mathfrak{C}g, \mathfrak{C}\eta, \mathfrak{C}\varepsilon$ form an adjunction between simplicial categories. In this case, there is a zig-zag of weak homotopy equivalences between $\mathcal{D}(fc, d)$ and $\mathcal{C}(c, gd)$ by Theorem 3.15.

Proposition 3.18. *Suppose $f: \mathcal{C} \rightarrow \mathcal{D}$ and $g: \mathcal{D} \rightarrow \mathcal{C}$ are adjoint functors between quasicategories with unit $\eta: 1_{\mathcal{C}} \Rightarrow gf$ and counit $\varepsilon: fg \Rightarrow 1_{\mathcal{D}}$. Then the right adjoint g is fully faithful if and only if*

every component of the counit ε is an equivalence. Dually, the left adjoint f is fully faithful if and only if every component of the unit η is an equivalence.

Proof. The functor g is fully faithful if and only if $\mathfrak{C}g$ is fully faithful by Theorem 3.15, which is the case if and only if the components $\mathfrak{C}\varepsilon$ are equivalences by Proposition 3.13. But the components of $\mathfrak{C}\varepsilon$ are the components of ε , so g is fully faithful if and only if the components of ε are equivalences. \square

Like the notion of adjunction between quasicategories, the notion of equivalence between quasicategories may also be phrased in terms of the 2-category \mathbf{SSet}^{τ_1} . A functor $f: \mathcal{C} \rightarrow \mathcal{D}$ between quasicategories is an *equivalence* if it is an equivalence in the 2-category \mathbf{SSet}^{τ_1} . That is to say, if there exist a functor $g: \mathcal{D} \rightarrow \mathcal{C}$, a morphism $\eta: \text{Id}_{\mathcal{C}} \rightarrow g \circ f$ in $\text{Fun}(\mathcal{C}, \mathcal{C})$, and a morphism $\varepsilon: f \circ g \rightarrow \text{Id}_{\mathcal{D}}$ in $\text{Fun}(\mathcal{D}, \mathcal{D})$ which induce invertible 2-cells $[\eta]$ and $[\varepsilon]$ in \mathbf{SSet}^{τ_1} . In other words, η and ε are natural equivalences. Every equivalence between quasicategories is a weak homotopy equivalence between the underlying simplicial sets.

A functor $f: \mathcal{C} \rightarrow \mathcal{D}$ between quasicategories is said to be *fully faithful* if the associated map $\mathcal{C}(a, b) \rightarrow \mathcal{D}(fa, fb)$ is a weak homotopy equivalence for all objects a and b of \mathcal{C} , recall $\mathcal{C}(a, b)$ is defined via the pullback as in equation (3.2). The functor f is said to be *essentially surjective* if $\tau_1 f: \tau_1 \mathcal{C} \rightarrow \tau_1 \mathcal{D}$ is essentially surjective.

Proposition 3.19. *Any equivalence of quasicategories is fully faithful and essentially surjective.*

Proof. Suppose f is an equivalence of quasicategories. Then $\mathfrak{C}f$ is an equivalence in $\mathbf{SimpCat}$ by the remarks at the end of Section 3.2, and $\mathfrak{C}f$ is fully faithful by Proposition 3.14. Then f is fully faithful by Theorem 3.15. For essential surjectivity of f , we note that τ_1 is a 2-functor as in Proposition 3.16, or we use the fact that $\pi_0: \mathbf{SimpCat} \rightarrow \mathbf{Cat}$ is a 2-functor so $\pi_0 \mathfrak{C}f = \tau_1 f$ is an equivalence because $\mathfrak{C}f$ is. \square

Also by Theorem 3.15 and the equality $\pi_0 \mathfrak{C} = \tau_1$, a functor f of quasicategories is fully faithful and essentially surjective if and only if $\mathfrak{C}f$ is fully faithful and essentially surjective.

To prove the converse of Proposition 3.19, we need to use some results about model structures for the first time in this paper.

Proposition 3.20. *Any fully faithful essentially surjective functor $f: X \rightarrow Y$ between quasicategories is an equivalence in \mathbf{SSet}^{τ_1} .*

Proof. Since f is fully faithful and essentially surjective, $\mathfrak{C}f$ is a Dwyer-Kan equivalence, so f is a weak equivalence in the Joyal model structure on \mathbf{SSet} , see [20, Theorem 6.12, page 299] and [9, Proposition 8.1]. In the Joyal model structure, all simplicial sets are cofibrant and the fibrant objects are precisely the quasicategories, so the weak equivalence

$f: X \rightarrow Y$ is a homotopy equivalence in the Joyal model structure [17, Proposition 1.2.8, page 11].

But by [20, Proposition 6.18, page 301], $X \times J[1]$ and $Y \times J[1]$ are cylinder objects on X and Y . Since f is a homotopy equivalence, there exists a functor $g: Y \rightarrow X$ and “homotopies” $X \times J[1] \rightarrow X$ and $Y \times J[1] \rightarrow Y$ from gf and fg to the respective identities. But these homotopies are precisely natural equivalences by Corollary 3.7, which also means they are iso 2-cells in the 2-category \mathbf{SSet}^{τ_1} . Hence f is an equivalence in \mathbf{SSet}^{τ_1} . \square

Every equivalence of quasicategories $f: X \rightarrow Y$ induces an ordinary equivalence of categories $\tau_1 f: \tau_1 X \rightarrow \tau_1 Y$. Consequently, if σ is a 1-morphism in X , then σ is an equivalence in X if and only if $f(\sigma)$ is an equivalence in Y . That is, every equivalence of quasicategories both preserves and reflects equivalences *in* quasicategories.

It is well known that every equivalence in a 2-category is part of an adjoint equivalence. Thus, if f is an equivalence of quasicategories, then g , η , and ε may be chosen to make f a left or right adjoint.

3.4. Commutative Squares and Pushouts in a Quasicategory.

Even though a quasicategory X is not equipped with a choice of composition, we may still speak of commutative squares in X , which are actually homotopy commutative squares in X . A *commutative square* in a quasicategory X is a functor $\Delta[1] \times \Delta[1] \rightarrow X$. Since τ_1 preserves products, a commutative square in X gives rise to a truly commutative square in the homotopy category $\tau_1 X$. The converse is also true.

Lemma 3.21. *Let X be a quasicategory. Every commutative square in its homotopy category $\tau_1 X$ comes from a commutative square in X .*

Proof. Since X is a quasicategory, every morphism in $\tau_1 X$ is a homotopy class $[f]$ of a morphism f in X . Moreover, we have $[g][f] = [h]$ in $\tau_1 X$ if and only if the boundary $\partial\Delta[2] \rightarrow X$ determined by g , f , and h extends to a functor $\Delta[2] \rightarrow X$, see [20, pages 212–213] which reference Boardman–Vogt [5]. Thus, if we have a commutative square $[g][f] = [k][j]$ in $\tau_1 X$, there is an $[h]$ in $\tau_1 X$ equal to both $[g][f]$ and $[k][j]$, and the corresponding two maps $\partial\Delta[2] \rightarrow X$ can be filled. Since $\Delta[1] \times \Delta[1]$ is 2-skeletal, this defines a functor $\Delta[1] \times \Delta[1] \rightarrow X$, which in turn induces the commutative square $[g][f] = [k][j]$. \square

For each natural transformation $\alpha: X \times \Delta[1] \rightarrow Y$ and each $f \in X_1$, the usual naturality square is a commutative square in Y .

We next recall the notion of pushout in a quasicategory X . Instead of recalling the general definition of colimit in a quasicategory from [19, Definition 4.5] and [20, page 159], we work out pushouts explicitly.

An object i of a quasicategory X is *initial* if for any object x in X the map $X(i, x) \rightarrow \text{pt}$ is a weak homotopy equivalence. If i is initial in

the quasicategory X , then it is initial in the category $\tau_1 X$ in the usual sense because

$$\text{pt} = \pi_0 X(i, x) = \pi_0 ((\mathfrak{C}X)(i, x)) = (\pi_0 \mathfrak{C}X)(i, x) = (\tau_1 X)(i, x)$$

by Theorem 3.15 and $\pi_0 \mathfrak{C} = \tau_1$. Any two initial objects i and i' of X are equivalent: the simplicial sets $X(i, i')$, $X(i', i)$, $X(i, i)$, $X(i', i')$ are all weakly equivalent to a point, so every $f \in X(i, i')$ and $g \in X(i', i)$ satisfy $[g][f] = [\text{Id}_i]$ and $[f][g] = [\text{Id}_{i'}]$ as $\pi_0 X(x, y) = (\tau_1 X)(x, y)$. Moreover, the 0-full subquasicategory of X on the initial objects is either a contractible Kan complex or the empty simplicial set by [20, page 159] or [21, Proposition 1.2.12.9].

If C is a category and i is an object of C , then i is initial in the category C if and only if i is initial in the quasicategory NC , as $(NC)(i, x)$ is $C(i, x)$ viewed as a discrete simplicial set.

Let K be the nerve of the category $b \leftarrow a \rightarrow c$ and $F: K \rightarrow X$ a functor. The quasicategory of commutative squares in X which restrict to F is the following pullback $X_{F/}$ in **SSet**.

$$(3.22) \quad \begin{array}{ccc} X_{F/} & \longrightarrow & X^{\Delta[1] \times \Delta[1]} \\ \downarrow & \text{pullback} & \downarrow \text{incl}^* \\ * & \xrightarrow{F} & X^K \end{array}$$

A commutative square in X is a *pushout square of the diagram $F: K \rightarrow X$* if it is an initial object in the quasicategory $X_{F/}$. Since initial objects are unique up to equivalence, any two pushout squares of F are equivalent. In particular, the lower right corner objects of two pushout squares of F are equivalent. Moreover, the 0-full subquasicategory of $X_{F/}$ on the pushouts (initial objects) is either a contractible Kan complex or the empty simplicial set.

In general, pushouts in X are *not* the same as ordinary pushouts in $\tau_1 X$. However pushouts in the nerve of a category NC are the same as pushouts in the category C . This is because: pushouts in C are defined precisely as initial objects in the categorical analogue of the pullback in (3.22), nerve preserves pullbacks, nerve is fully faithful, $NC^{ND} \cong N(C^D)$ for any category D , in particular for $[1] \times [1]$ and $b \leftarrow a \rightarrow c$, and initial objects in a category are the same as initial objects in its nerve.

Though we do not need this, we remark that if \mathcal{C} is a category enriched in Kan complexes, then homotopy pushouts in \mathcal{C} correspond to pushouts in $N^{\text{simp}} \mathcal{C}$, see [21, Theorem 4.2.4.1, page 258].

In an ordinary category, the pushout of any isomorphism along any morphism exists and is also an isomorphism. We have the following analogue in a quasicategory.

Lemma 3.23. *A pushout of any equivalence along any morphism in a quasicategory X exists, and is an equivalence in X .*

Proof. (Sketch) Suppose $j: A \rightarrow C$ is an equivalence in X and $f: A \rightarrow B$ is any morphism. Let fj^{-1} denotes any filler for the horn determined by f and any pseudo inverse to the equivalence j . Then the outer square pictured below is a pushout.

Suppose the inner square below is a pushout square in X .

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 j \downarrow & \text{pushout} & \downarrow j' \\
 C & \xrightarrow{f'} & P \\
 & \searrow f'j^{-1} & \downarrow m \\
 & & B
 \end{array}$$

(Note: The diagram shows a square with vertices A, B, C, P. A horizontal arrow f goes from A to B. A vertical arrow j goes from A to C. A vertical arrow j' goes from B to P. A horizontal arrow f' goes from C to P. A curved arrow fj^{-1} goes from C to B. A dashed arrow m goes from P to B. A curved arrow Id_B goes from B to B. The text 'pushout' is between the vertical arrows j and j'. The text 'Id_B' is next to the curved arrow from B to B.)

Since pushouts are unique up to equivalence, there is an equivalence m which makes the relevant triangles commute in the homotopy category of X . By the 3-for-2 property of equivalences, j' is then also an equivalence. \square

4. ADDITIVITY FOR S_\bullet^∞ OF A QUASICATEGORY

We prove the additivity of S_\bullet^∞ , the quasicategorical version of Waldhausen's S_\bullet construction [32, 1.3, pages 328-329].

Definition 4.1 (Waldhausen quasicategory). A *Waldhausen quasicategory* consists of a quasicategory \mathcal{C} together with a distinguished zero object $*$ and a subquasicategory $co\mathcal{C}$, the 1-simplices of which are called *cofibrations* and denoted \rightarrowtail , such that

- (i) The subquasicategory $co\mathcal{C}$ is 1-full in \mathcal{C} and contains all equivalences in \mathcal{C} ,
- (ii) For each object A of \mathcal{C} , every morphism $* \rightarrow A$ is a cofibration,
- (iii) The pushout of a cofibration along any morphism exists, and every pushout of a cofibration along any morphism is a cofibration.

Several remarks about the definition of Waldhausen quasicategory are in order. The subquasicategory $co\mathcal{C}$ contains $\mathcal{C}_{\text{equiv}}$ by (i) because $\mathcal{C}_{\text{equiv}}$ is 1-full on the equivalences. The definition of Waldhausen quasicategory does not require an additional structure $w\mathcal{C}$ because every quasicategory \mathcal{C} already has a subquasicategory of equivalences, namely $\mathcal{C}_{\text{equiv}}$. Nor is a “gluing lemma” required (pushouts in a quasicategory are automatically invariant under equivalence).

Barwick’s notion of Waldhausen ∞ -category in [1, Definition 2.4] implies (i), (ii), (iii) (though he does not distinguish a zero object). The

main difference is that instead of a 1-full subquasicategory of cofibrations, he requires a “subcategory” [21, Section 1.2.11] of cofibrations, which implies 1-fullness. His requirement in (2.4.3) that the source functor be a cocartesian fibration is equivalent to (iii) above by the dual to [21, Lemma 6.1.1.1], as he remarks in 2.5. Equivalences are cofibrations by his notion of pair in Definition 1.11.

Another observation from the definition of Waldhausen quasicategory is that $\tau_1(\text{co}\mathcal{C})$ is naturally a subcategory of $\tau_1\mathcal{C}$ by 1-fullness and (iii). Namely, recall that a subquasicategory \mathcal{R} of a quasicategory X is *homotopy replete* if for every commutative square in X with vertical maps weak equivalences

$$(4.2) \quad \begin{array}{ccc} x & \xrightarrow{r} & y \\ \text{equiv} \downarrow & & \downarrow \text{equiv} \\ x' & \xrightarrow{r'} & y' \end{array}$$

we have $r \in \mathcal{R} \Leftrightarrow r' \in \mathcal{R}$, see [20, Definition F.1.1] for this definition a model category. If \mathcal{R} is homotopy replete in X and f is a morphism in \mathcal{R} , then any morphism g homotopic to f is also in \mathcal{R} .

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ 1_x \downarrow & \searrow f & \downarrow 1_y \\ x & \xrightarrow{g} & y \end{array}$$

In a Waldhausen quasicategory, $\text{co}\mathcal{C}$ is homotopy replete by (iii) because any commutative square of the form (4.2) is a pushout. So the homotopy class of a cofibration in $\text{co}\mathcal{C}$ is the same as its homotopy class in \mathcal{C} . By 1-fullness of $\text{co}\mathcal{C}$, a relation $[g][f] = [h]$ holds in $\tau_1(\text{co}\mathcal{C})$ if and only if it holds in $\tau_1(\mathcal{C})$, and we now have $\tau_1(\text{co}\mathcal{C})$ naturally embedded in $\tau_1(\mathcal{C})$.

Example 4.3. If \mathcal{C} is a (classical) Waldhausen category in which the weak equivalences are the isomorphisms, then $\text{co}(N\mathcal{C}) := N(\text{co}\mathcal{C})$ makes $N\mathcal{C}$ into a Waldhausen quasicategory. Pushouts in $N\mathcal{C}$ are the same as pushouts in \mathcal{C} .

Example 4.4 (D. Gepner). If \mathcal{C} is a (classical) Waldhausen category in which the weak equivalences are not the isomorphisms, then one may construct a Waldhausen quasicategory \mathcal{C}' with the same K -theory as \mathcal{C} , though this Waldhausen quasicategory is not simply $N\mathcal{C}$. Let \mathcal{C}' be the localization⁷ of $N\mathcal{C}$ with respect to $N(w\mathcal{C})$, and let $\text{co}\mathcal{C}'$ be the smallest 1-full subquasicategory of \mathcal{C}' which contains the cofibrations

⁷See [20, page 168]. Here this means the localization quasicategory \mathcal{C}' in the sense of a map $N\mathcal{C} \rightarrow \mathcal{C}'$ which takes the morphisms of $w\mathcal{C}$ to equivalences in \mathcal{C}' in a universal way. This does not require the stronger sense of localization in terms of a fully faithful right adjoint as Lurie does in [21, Section 5.2.7].

of \mathcal{C} and the equivalences of \mathcal{C}' . A morphism of \mathcal{C}' is a concatenation of morphisms in \mathcal{C} and formally inverted morphisms of $w\mathcal{C}$, while a 1-morphism of $co\mathcal{C}'$ is a concatenation of cofibrations in \mathcal{C} and formally inverted morphisms of $w\mathcal{C}$ (to see this, one uses that fact that $\tau_1\mathcal{C}'$ is an equivalent category to $(\tau_1N\mathcal{C})[(Nw\mathcal{C})^{-1}]$). By construction, axioms (i) and (ii) of Definition 4.1 hold. For (iii), if we have a diagram of the form $\bullet \leftarrow \bullet \rightrightarrows \bullet$ then we can construct an $m \times n$ grid of pushout squares, one row at a time, using pushouts along cofibrations in \mathcal{C} and Lemma 3.23. A composite exists and is a pushout square. The composite edge across from the cofibration is also a cofibration by the description above. The K -theory of \mathcal{C}' , which we define below, is the same as the K -theory of \mathcal{C} by a quasicategorical analogue of Waldhausen's Fibration Theorem.

Example 4.5. Any *stable* quasicategory is a Waldhausen quasicategory when all morphisms are considered cofibrations. For instance, the quasicategory of modules for an A_∞ ring spectrum R , or the quasicategory of complexes of quasicoherent \mathcal{O}_X -modules over a quasicompact and quasiseparated scheme X .

We include cofibrations in the definition of $[n]$ -complex, $\text{Gap}([n], \mathcal{C})$, and $S_\bullet^\infty \mathcal{C}$ from 1.2.2.2 and 1.2.2.5 of Lurie [22] below. See Blumberg–Gepner–Tabuada [2, Section 6.1] for a comparison of S_\bullet^∞ with the S'_\bullet construction of [3] in the case of a simplicial model category which admits all finite homotopy colimits (all maps in a model category are weak cofibrations, so in [3] the cofibrations do not play a role in the application of S_\bullet^∞ to the simplicial nerve of the full sub-simplicial category on the fibrant-cofibrant objects).

Definition 4.6 (S_\bullet^∞ construction). Let \mathcal{C} be a Waldhausen quasicategory. An $[n]$ -complex is a map of simplicial sets $F: N\text{Ar}[n] \rightarrow \mathcal{C}$ such that

- (i) For each $i \in [n]$, $F(i, i)$ is the distinguished zero object of \mathcal{C} ,
- (ii) For each $i \leq j \leq k$, the morphism $F(i, j) \rightarrow F(i, k)$ is a cofibration,
- (iii) For each $i \leq j \leq k$, the diagram

$$\begin{array}{ccc} F(i, j) & \twoheadrightarrow & F(i, k) \\ \downarrow & & \downarrow \\ F(j, j) & \twoheadrightarrow & F(j, k) \end{array}$$

is a pushout square in \mathcal{C} .

Let $\text{Gap}([n], \mathcal{C})$ be the full sub-simplicial set of $\text{Map}(N\text{Ar}[n], \mathcal{C})$ with vertices the $[n]$ -complexes. The S_\bullet^∞ construction of \mathcal{C} is the simplicial quasicategory $S_\bullet^\infty \mathcal{C}$ defined by

$$S_n^\infty \mathcal{C} = \text{Gap}([n], \mathcal{C}) \subseteq \mathbf{S}\mathbf{Set}(N(\text{Ar}[n]) \times \Delta[-], \mathcal{C}).$$

The face and degeneracy maps of the simplicial object $S_\bullet^\infty \mathcal{C}$ in the category \mathbf{QCat} are induced from Δ via the category of arrows construction.

Each simplicial set $S_n^\infty \mathcal{C}$ is a quasicategory, since $\text{Fun}(N\text{Ar}[n], \mathcal{C}) = \mathcal{C}^{N\text{Ar}[n]}$ is a quasicategory, and $S_n^\infty \mathcal{C}$ is full on its set of vertices. If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor of quasicategories, then the functor $S_n^\infty F: S_n^\infty \mathcal{C} \rightarrow S_n^\infty \mathcal{D}$ is the restriction of the functor $F_*: \text{Fun}(N\text{Ar}[n], \mathcal{C}) \rightarrow \text{Fun}(N\text{Ar}[n], \mathcal{D})$. If $\alpha: \mathcal{C} \times \Delta[1] \rightarrow \mathcal{D}$ is a natural transformation, then $S_n^\infty \alpha$ is the natural transformation with level m given by the restriction of

$$\mathbf{SSet}(N\text{Ar}[n] \times \Delta[m], \mathcal{C}) \times \Delta[1]_m \longrightarrow \mathbf{SSet}(N\text{Ar}[n] \times \Delta[m], \mathcal{D})$$

$$(f, j) \longmapsto \alpha \circ (f, j_* \text{pr}_2) .$$

If α is a natural equivalence, then so is $S_n^\infty \alpha$.

Each simplicial set $S_n^\infty \mathcal{C}$ is even a Waldhausen quasicategory, and as a consequence of Additivity Theorem 4.11 there is an Ω -spectrum (beyond the 0-th term) with

$$K(\mathcal{C})_n := | (S_\bullet^\infty \cdots S_\bullet^\infty \mathcal{C})_{\text{equiv}} |$$

for $n \geq 0$, where S_\bullet appears n times.

Example 4.7. If \mathcal{C} is a (classical) Waldhausen category in which the weak equivalences are the isomorphisms, then $co(N\mathcal{C}) = N(co\mathcal{C})$ and $NwS_n \mathcal{C} \cong (S_n^\infty N\mathcal{C})_{\text{equiv}}$ so that $K(N\mathcal{C}) \cong K(\mathcal{C})$. For the proof, first note $N(S_n \mathcal{C}) \cong S_n^\infty N\mathcal{C}$ because

$$\begin{aligned} \mathbf{Cat}(\text{Ar}[n] \times [m], \mathcal{C}) &\cong \mathbf{SSet}(N(\text{Ar}[n] \times [m]), N\mathcal{C}) \\ &\cong \mathbf{SSet}(N(\text{Ar}[n]) \times \Delta[m], N\mathcal{C}) \end{aligned}$$

and pushout squares in $N\mathcal{C}$ are the same as pushout squares in \mathcal{C} .

Then we also have by Lemma 3.4

$$\begin{aligned} NwS_n \mathcal{C} &= N((S_n \mathcal{C})_{\text{iso}}) \\ &= (N(S_n \mathcal{C}))_{\text{equiv}} \\ &\cong (S_n^\infty N\mathcal{C})_{\text{equiv}} . \end{aligned}$$

Moving towards Additivity, we now consider the quasicategory of cofiber sequences with subobject and quotient in specified subquasicategories.

Notation 4.8. Recall Example 2.6. Let \mathcal{C} be a Waldhausen quasicategory and \mathcal{A} and \mathcal{B} sub Waldhausen quasicategories. Let s' , q' , and p'_{01} be the inclusions of $\{(0, 0)\}$, $\{(1, 1)\}$, and $\{(0, 1)\}$ into $[1] \times [1]$ respectively. These induce “subobject”, “quotient”, and “projection to lower left corner” $s, q, p: \mathcal{C}^{\Delta[1] \times \Delta[1]} \rightarrow \mathcal{C}$, which on an n -simplex

$\sigma: \Delta[1] \times \Delta[1] \times \Delta[n] \rightarrow \mathcal{C}$ are defined as

$$\begin{aligned} s(\sigma) &= \sigma(s', -) \\ q(\sigma) &= \sigma(q', -) \\ p(\sigma) &= \sigma(p', -). \end{aligned}$$

The quasicategory $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is the following pullback in \mathbf{SSet} .

$$\begin{array}{ccc} \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) & \longrightarrow & (\mathcal{C}^{\Delta[1] \times \Delta[1]})_{\text{pushout}} \\ \downarrow & \text{pullback} & \downarrow (s, q, p) \\ \mathcal{A} \times \mathcal{B} \times * & \xrightarrow{\text{incl}} & \mathcal{C} \times \mathcal{C} \times \mathcal{C} \end{array}$$

Here $(\mathcal{C}^{\Delta[1] \times \Delta[1]})_{\text{pushout}}$ is the subquasicategory of $\mathcal{C}^{\Delta[1] \times \Delta[1]}$ 0-full on the pushout squares in \mathcal{C} . The quasicategory $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is the *quasicategory of cofiber sequences in \mathcal{C} of the form $A \rightarrowtail C \twoheadrightarrow B$ with $A \in \mathcal{A}_0$ and $B \in \mathcal{B}_0$* . An n -simplex of $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is a map $\sigma: \Delta[1] \times \Delta[1] \times \Delta[n] \rightarrow \mathcal{C}$ such that $\sigma(0, 0, -)$ goes into \mathcal{A} , $\sigma(1, 1, -)$ goes into \mathcal{B} , for all $0 \leq \ell \leq n$ we have $\sigma(1, 0, \ell) = *$, and each square $\sigma(-, -, \ell)$ is a pushout square.

A morphism in $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is a *cofibration* if any (every) “induced map” $C_1 \cup_{A_1} A_2 \rightarrow C_2$ is a cofibration in \mathcal{C} , (this is well defined because any map homotopic to a cofibration is a cofibration). Then $\text{co}\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is defined to be the smallest 1-full homotopy replete subquasicategory containing these cofibrations and the equivalences.

In his proof of Additivity, Waldhausen uses the object part \mathfrak{s}_\bullet of the S_\bullet construction, as we recalled in Section 1. We will do the same in the quasicategory context, where it means to truncate to the 0-part.

Definition 4.9 (The simplicial set $\mathfrak{s}_\bullet^\infty \mathcal{C}$). Let \mathcal{C} be a Waldhausen quasicategory. The simplicial set $\mathfrak{s}_\bullet^\infty \mathcal{C}$ is the object part of the simplicial quasicategory $S_\bullet^\infty \mathcal{C}$, that is

$$\mathfrak{s}_n^\infty \mathcal{C} = (S_n^\infty \mathcal{C})_0 = \text{Gap}([n], \mathcal{C})_0 \subseteq \mathbf{SSet}(N\text{Ar}[n], \mathcal{C}).$$

Lemma 4.10. *Let \mathcal{C} be a Waldhausen quasicategory. Then the map of simplicial sets*

$$\mathfrak{s}_\bullet^\infty(s, q): \mathfrak{s}_\bullet^\infty \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \longrightarrow \mathfrak{s}_\bullet^\infty \mathcal{A} \times \mathfrak{s}_\bullet^\infty \mathcal{B}$$

is a weak equivalence.

Proof. The proof of Lemma 1.9 was entirely simplicial except Footnote 2, so the same argument works here, provided we justify Footnote 2. To work with the same diagrams as in (1.10), we only need to notice that $\mathfrak{s}_n^\infty \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \cong \mathcal{E}(S_n^\infty \mathcal{A}, S_n^\infty \mathcal{C}, S_n^\infty \mathcal{B})_0$ as simplicial sets.

Concerning Footnote 2, let \mathcal{I} be the category $\bullet \leftarrow \bullet \rightarrow \bullet$ and \mathcal{J} the category consisting of the free standing commutative square. Let $(\mathcal{C}^{N\mathcal{J}})_{\text{poc}}$ denote the 0-full subquasicategory of $\mathcal{C}^{N\mathcal{J}}$ on the pushout squares in \mathcal{C} in which the first leg is a cofibration. Let $(\mathcal{C}^{N\mathcal{I}})_c$ denote

the 0-full subquasicategory of $\mathcal{C}^{N\mathcal{I}}$ on diagrams in which the first leg is a cofibration, and consider the restriction map $(\mathcal{C}^{N\mathcal{J}})_{\text{poc}} \rightarrow (\mathcal{C}^{N\mathcal{I}})_{\text{c}}$.

The fibers of this restriction are contractible Kan complexes, since a fiber is the 0-full subcategory on the initial objects in a slice quasicategory, and pushouts along cofibrations exist by hypothesis.

This restriction is a trivial Kan fibration, so it admits a section. We fix a section, and this is our coherent choice of pushouts in \mathcal{C} . \square

Theorem 4.11 (Waldhausen Additivity for Quasicategories). *Let \mathcal{C} be a Waldhausen quasicategory and \mathcal{A} and \mathcal{B} sub Waldhausen quasicategories. Then (s, q) induces a diagonal weak equivalence*

$$S_{\bullet}^{\infty}(s, q)_{\text{equiv}} : (S_{\bullet}^{\infty}\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}))_{\text{equiv}} \longrightarrow (S_{\bullet}^{\infty}\mathcal{A})_{\text{equiv}} \times (S_{\bullet}^{\infty}\mathcal{B})_{\text{equiv}}$$

of bisimplicial sets.

Proof. We use the weak equivalence on the object simplicial sets in Lemma 4.10 in a way similar to [32, page 336]. Recall that $J[m]$ denotes the nerve of the groupoid with objects $0, 1, \dots, m$ and a unique isomorphism from any object to another. By Lemma 4.10, the map of simplicial sets

$$\mathfrak{s}_{\bullet}^{\infty}\mathcal{E}(\mathcal{A}^{J[m]}, \mathcal{C}^{J[m]}, \mathcal{B}^{J[m]}) \longrightarrow \mathfrak{s}_{\bullet}^{\infty}(\mathcal{A}^{J[m]}) \times \mathfrak{s}_{\bullet}^{\infty}(\mathcal{B}^{J[m]})$$

is a weak equivalence. Then by the Realization Lemma, (4.13)

$$([m] \mapsto \mathfrak{s}_{\bullet}^{\infty}\mathcal{E}(\mathcal{A}^{J[m]}, \mathcal{C}^{J[m]}, \mathcal{B}^{J[m]})) \longrightarrow ([m] \mapsto \mathfrak{s}_{\bullet}^{\infty}(\mathcal{A}^{J[m]})) \times ([m] \mapsto \mathfrak{s}_{\bullet}^{\infty}(\mathcal{B}^{J[m]}))$$

is a weak equivalence of simplicial simplicial sets.

We prove $\mathfrak{s}_{\bullet}^{\infty}(\mathcal{D}^{J[-]}) \cong (S_{\bullet}^{\infty}\mathcal{D})_{\text{equiv}}$ for any Waldhausen quasicategory \mathcal{D} , and for this we use Corollary 3.6.

$$\begin{aligned} \mathfrak{s}_n^{\infty}(\mathcal{D}^{J[m]}) &= \text{Gap}([n], \mathcal{D}^{J[m]})_0 \\ &\subseteq \text{Map}(N\text{Ar}[n], \mathcal{D}^{J[m]})_0 \\ &= \mathbf{S}\text{Set}(N\text{Ar}[n], \mathcal{D}^{J[m]}) \\ &\cong \mathbf{S}\text{Set}(J[m], \mathcal{D}^{N\text{Ar}[n]}) \\ &\supseteq \mathbf{S}\text{Set}(J[m], \text{Gap}([n], \mathcal{D})) \\ &= (\text{Gap}([n], \mathcal{D})_{\text{equiv}})_m \\ &= ((S_n^{\infty}\mathcal{D})_{\text{equiv}})_m \end{aligned}$$

The isomorphism in the middle preserves $[n]$ -complexes.

By a similar argument, $\mathcal{E}(\mathcal{A}^{J[m]}, \mathcal{C}^{J[m]}, \mathcal{B}^{J[m]}) \cong \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})^{J[m]}$. The map (4.12) and the weak equivalence (4.13) now fit into a commutative square in which three of the four maps are diagonal weak equivalences of bisimplicial sets. Hence (4.12) is also a weak equivalence. \square

5. QUASICATEGORICAL ADDITIVITY AND SPLIT-EXACT SEQUENCES OF WALDHAUSEN QUASICATEGORIES

Definition 5.1 (Exact functor). Let \mathcal{C} and \mathcal{D} be Waldhausen quasicategories. A functor $f: \mathcal{C} \rightarrow \mathcal{D}$ is called *exact* if $f(*) = *$, $f(\text{co}\mathcal{C}) \subseteq \text{co}\mathcal{D}$, and f maps each pushout square along a cofibration to a pushout square along a cofibration.

Note that by 1-fullness of the subquasicategory of cofibrations, it suffices to check that f maps cofibrations to cofibrations to conclude that $f(\text{co}\mathcal{C}) \subseteq \text{co}\mathcal{D}$. Note also that every functor of quasicategories maps equivalences to equivalences, so $f(\mathcal{C}_{\text{equiv}}) \subseteq \mathcal{D}_{\text{equiv}}$ by 1-fullness, so there is no need to require that in the definition of exact functor.

Definition 5.2 (Exact sequence of Waldhausen quasicategories). Let \mathcal{A} , \mathcal{E} , and \mathcal{B} be Waldhausen quasicategories. A sequence of exact morphisms

$$(5.3) \quad \mathcal{A} \xrightarrow{i} \mathcal{E} \xrightarrow{f} \mathcal{B}$$

is called *exact* if

- (i) the composite $f \circ i$ is the distinguished zero object $*$ of \mathcal{B} ,
- (ii) the exact morphism $i: \mathcal{A} \rightarrow \mathcal{E}$ is fully faithful, and
- (iii) the restricted morphism $f|_{\mathcal{E}/\mathcal{A}}: \mathcal{E}/\mathcal{A} \rightarrow \mathcal{B}$ is an equivalence of quasicategories. Here \mathcal{E}/\mathcal{A} is the 0-full subquasicategory of \mathcal{E} on the objects $E \in \mathcal{E}$ such that the simplicial set $\mathcal{E}(i(A), E)$ is weakly contractible for all $A \in \mathcal{A}$.

Definition 5.4 (Split-exact sequence of Waldhausen quasicategories). An exact sequence of Waldhausen quasicategories as in equation (5.3) is called *split* if there exist exact morphisms

$$\mathcal{A} \xleftarrow{j} \mathcal{E} \xleftarrow{g} \mathcal{B}$$

right adjoint to i and f respectively, such that the unit $\text{Id}_{\mathcal{A}} \rightarrow ji$ and the counit $fg \rightarrow \text{Id}_{\mathcal{B}}$ are natural equivalences.

Remark 5.5. In a split-exact sequence of Waldhausen quasicategories and exact functors, the functor g is actually an inverse equivalence to $f|_{\mathcal{E}/\mathcal{A}}$. We first see that g goes into \mathcal{E}/\mathcal{A} , as there is a zig-zag of weak homotopy equivalences between $\mathcal{E}(i(A), g(B))$ and $\mathcal{B}(f(i(A)), B) = \mathcal{B}(*, B)$, which is weakly equivalent to a point. The counit $(f|_{\mathcal{E}/\mathcal{A}})g \rightarrow \text{Id}_{\mathcal{B}}$ is a natural equivalence by hypothesis, and the unit $\text{Id}_{\mathcal{E}/\mathcal{A}} \rightarrow g(f|_{\mathcal{E}/\mathcal{A}})$ is a natural equivalence since the left adjoint f is fully faithful by Proposition 3.18 and Proposition 3.19.

Example 5.6. Any Waldhausen quasicategory \mathcal{C} with selected sub Waldhausen quasicategories \mathcal{A} and \mathcal{B} produces a split-exact sequence

$$(5.7) \quad \mathcal{A} \xrightarrow{i} \mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B}) \xrightarrow{q} \mathcal{B}$$

$\xleftarrow{s} \quad \quad \quad \xleftarrow{g}$

of Waldhausen quasicategories as follows. Recall Notation 4.8 in which $\mathcal{E}(\mathcal{A}, \mathcal{C}, \mathcal{B})$, s , and q are defined for quasicategories. Fix two natural transformations $\alpha, \beta: \Delta[1] \times \mathcal{C} \rightarrow \mathcal{C}$ with $\alpha_0 = \text{Id}_{\mathcal{C}}$ and $\alpha_1 = *$, and $\beta_0 = *$ and $\beta_1 = \text{Id}_{\mathcal{C}}$. For $A: \Delta[n] \rightarrow \mathcal{A}$ and $B: \Delta[n] \rightarrow \mathcal{B}$, we define $i(A)$ and $g(B)$ to be the composites

$$\begin{array}{ccc} \Delta[1] \times \Delta[1] \times \Delta[n] & \xrightarrow{\text{pr}_2 \times A} & \Delta[1] \times \mathcal{A} \xrightarrow{\alpha} \mathcal{C} \\ & \searrow^{i(A)} & \\ \Delta[1] \times \Delta[1] \times \Delta[n] & \xrightarrow{\text{pr}_1 \times B} & \Delta[1] \times \mathcal{B} \xrightarrow{\beta} \mathcal{C} \\ & \searrow_{g(B)} & \end{array}$$

Then clearly $q \circ i = *$. Also $\text{Id}_{\mathcal{A}} = si$, and $qg = \text{Id}_{\mathcal{B}}$, so the unit of the adjunction $i \dashv s$ and the counit of the adjunction $q \dashv g$ are the identity natural transformations so i and g are fully faithful (see Proposition 3.18). The counit and unit of these respective adjunctions can be defined as in Example 2.6, and the triangle identities are also proved similarly (recall the discussion of adjunctions for quasicategories in Section 3.3).

We next claim that the subcategory \mathcal{E}/\mathcal{A} is 0-full on the cofiber sequences equivalent to those of the form in equation (2.8) with $B \in \mathcal{B}$. Suppose $A' \rightarrow C' \rightarrow B'$ is a vertex of \mathcal{E}/\mathcal{A} . Then $\mathcal{E}(i(A), A'C'B')$ is weakly contractible, so $(\tau_1 \mathcal{E})(i(A), A'C'B') = \pi_0 \mathcal{E}(i(A), A'C'B')$ consists of a single point for every $A \in \mathcal{A}_0$. Consideration of diagram (2.9) in $\tau_1 \mathcal{E}$ shows that A' is a terminal object of $\tau_1 \mathcal{A}$ (note that the commutative squares in $\tau_1 \mathcal{E}$ induced by cofiber sequences are not necessarily pushouts in $\tau_1 \mathcal{E}$). But $*$ is also terminal in $\tau_1 \mathcal{A}$, so there is an isomorphism in $\tau_1 \mathcal{A}$ between A' and $*$, and an equivalence in \mathcal{A} between A' and $*$, so A' is also a zero object of \mathcal{A} . Finally, $C' \rightarrow B'$ is an equivalence, as it is a pushout in \mathcal{C} of an equivalence $A' \rightarrow *$. This equivalence $C' \rightarrow B'$ is part of an equivalence in \mathcal{E} from $A' \rightarrow C' \rightarrow B'$ to $* \rightarrow B' \rightarrow B'$, which is of the form (2.8). Conversely, $g(B) \in \mathcal{E}/\mathcal{A}$ because $\mathcal{E}(i(A), g(B)) \rightarrow \mathcal{B}(qi(A), B) = \mathcal{B}(*, B) \rightarrow \text{pt}$ is a weak equivalence.

We now know $g: \mathcal{B} \rightarrow \mathcal{E}/\mathcal{A}$ is essentially surjective and fully faithful. Consequently, g is an equivalence of quasicategories (see Proposition 3.20). By the uniqueness of adjoints in a 2-category (here \mathbf{SSet}^{τ_1}),

the left adjoint $q|_{\mathcal{E}/\mathcal{A}}: \mathcal{E}/\mathcal{A} \rightarrow \mathcal{B}$ to g is also an equivalence of quasiequivalences, and we have shown that (5.7) is a split-exact sequence of quasiequivalences.

Proposition 5.8. *Suppose a split-exact sequence of Waldhausen quasiequivalences and exact functors*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{f} & \mathcal{B} \\ & \nwarrow j & \nwarrow g & & \\ & & & & \end{array}$$

has the following three properties.

- (i) *Each counit component $ij(E) \rightarrow E$ is a cofibration.*
- (ii) *For each morphism $E \rightarrow E'$ in \mathcal{E} , the induced map*

$$E \cup_{ij(E)} ij(E') \rightarrow E'$$

is a cofibration in \mathcal{E} .

- (iii) *In every cofiber sequence in \mathcal{A} of the form $A_0 \rightarrowtail A_1 \twoheadrightarrow *$, the first map is an isomorphism.*

Then it is Waldhausen equivalent (Definition 6.8) to a split-exact sequence of the form (5.7) in Example 5.6.

Proof. Let $\mathcal{C} := \mathcal{E}$ and let $\mathcal{A}', \mathcal{B}' \subseteq \mathcal{C}$ be the subquasiequivalences of \mathcal{C} that are 0-full on objects equivalent to objects in the images of i and g respectively. Since the unit $\text{Id}_{\mathcal{A}} \rightarrow ji$ and counit $fg \rightarrow \text{Id}_{\mathcal{B}}$ are natural equivalences, the left adjoint i and the right adjoint g are fully faithful, and therefore provide equivalences with the essential images \mathcal{A}' and \mathcal{B}' as pictured in (2.11).

Fix two natural transformations $\alpha, \beta: \Delta[1] \times \mathcal{C} \rightarrow \mathcal{C}$ with $\alpha_0 = \text{Id}_{\mathcal{C}}$ and $\alpha_1 = *$, and $\beta_0 = *$ and $\beta_1 = \text{Id}_{\mathcal{C}}$. Anytime we draw an arrow to or from $*$, we mean the respective component of α or β . Also, fix for every two objects C and C' of \mathcal{C} a “composite” of these arrows $C \rightarrow * \rightarrow C'$. For any object $E \in \mathcal{C}$, define $\Phi(E)$ to be the commutative square

$$(5.9) \quad \begin{array}{ccc} ij(E) & \xrightarrow{\text{counit}} & E \\ \downarrow & \searrow & \downarrow \text{unit} \\ * & \xrightarrow{\quad} & gf(E) \end{array}$$

(this commutes because $\mathcal{E}(ij(E), gf(E))$ is weakly equivalent to a point, see Remark 5.5). If $\nu: \Delta[n] \rightarrow \mathcal{C}$ is an n -simplex, we define $\sigma = \Phi(\nu)$ by

$$\begin{aligned} \sigma(0, 0, -) &= ij\nu \\ \sigma(0, 1, -) &= \nu \\ \sigma(1, 0, -) &= * \\ \sigma(1, 1, -) &= gf\nu \\ \sigma(-, -, \ell) &= \text{diagram (5.9) for } \nu(\ell) \end{aligned}$$

for $0 \leq \ell \leq n$. This definition of Φ makes the quasicategorical version of 2.18 strictly commute.

We claim that (5.9) is a pushout in the quasicategory \mathcal{C} . Let P be a pushout of the counit $ij(E) \rightarrow E$ along $ij(E) \rightarrow *$, as pictured on the left of diagram (2.13). There is a (homotopically unique) natural transformation from the P -square to (5.9). Applying j to this natural transformation gives us the quasicategorical version of the right diagram in (2.13). Arguing as in that passage, using the 3-for-2 property of equivalences in \mathcal{C} and the fact that pushouts of equivalences are equivalences, we have $P \in \mathcal{E}/\mathcal{A}$. Then by Remark 5.5, there exists a $Q \in \mathcal{B}$ and an equivalence $P \rightarrow g(Q)$. Proceeding as in diagram (2.14), we obtain a natural equivalence between (5.9) and a pushout square.

One can also show that the functor Φ is fully faithful.

To show $\Phi: \mathcal{E} \rightarrow \mathcal{E}(\mathcal{A}', \mathcal{C}, \mathcal{B}')$ is essentially surjective it suffices to prove that the top row of the middle diagram in (2.15) is isomorphic in $\tau_1 \mathcal{E}$ to the bottom row of the middle diagram in (2.15) (every commutative cube in $\tau_1 \mathcal{E}$ comes from a “commutative” cube in \mathcal{E} as in Lemma 3.21, so we can extend the three isomorphisms to get a natural equivalence of squares in \mathcal{E}). Recall from Proposition 3.16 that τ_1 is a 2-functor, so that $i \dashv j$ and $f \dashv g$ induce ordinary adjunctions between the respective homotopy categories, and we can use the universality of the counit and unit on the level of homotopy categories. Therefore, morphisms $[m]$ and $[n]$ exist as in diagram (2.15). Then m and n are seen to be equivalences, arguing via applications of f and j on the quasicategory level as in the passage after (2.15) with the word “isomorphism” replaced by “equivalence” (this is where the hypothesis on \mathcal{A} is used).

Lastly, Φ is a Waldhausen equivalence of quasicategories by Proposition 6.9. \square

Theorem 5.10 (Quasicategorical Waldhausen Additivity, General and Spectral Forms). *Suppose a split-exact sequence of Waldhausen quasicategories and exact functors*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{i} & \mathcal{E} & \xrightarrow{f} & \mathcal{B} \\ & \searrow j & & \nwarrow g & \\ & & & & \end{array}$$

has the following three properties.

- (i) *Each counit component $ij(E) \rightarrow E$ is a cofibration.*
- (ii) *For each morphism $E \rightarrow E'$ in \mathcal{E} , the induced map*

$$E \cup_{ij(E)} ij(E') \rightarrow E'$$

is a cofibration in \mathcal{E} .

- (iii) *In every cofiber sequence in \mathcal{A} of the form $A_0 \rightarrow A_1 \rightarrow *$, the first map is an isomorphism.*

Then the following hold.

(i) *The map*

$$S_{\bullet}^{\infty}(j, f)_{\text{equiv}}: (S_{\bullet}^{\infty}\mathcal{E})_{\text{equiv}} \longrightarrow (S_{\bullet}^{\infty}\mathcal{A})_{\text{equiv}} \times (S_{\bullet}^{\infty}\mathcal{B})_{\text{equiv}}$$

is a diagonal weak equivalence of bisimplicial sets.

(ii) *The functors j and f induce a stable equivalence of K -theory spectra*

$$K(j, f): K(\mathcal{E}) \longrightarrow K(\mathcal{A}) \vee K(\mathcal{B}).$$

Proof. For (i), we observe that the quasicategorical version of diagram (2.18) also strictly commutes, then we use Proposition 5.8, Corollary 6.11, Theorem 4.11, and the 3-for-2 property of weak homotopy equivalences of spaces.

The proof of (ii) is the same as the proof of Theorem 2.20. \square

6. APPENDIX: EQUIVALENCES OF WALDHAUSEN (QUASI)CATEGORIES

One detail we need in the proof of General Waldhausen Additivity, Theorem 2.19, is that Waldhausen equivalences induce weak homotopy equivalences after application of wS_{\bullet} . We prove this for Waldhausen categories now, and then turn to the easier proof for Waldhausen quasicategories.

Definition 6.1. An exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between Waldhausen categories is a *Waldhausen equivalence* if there is an exact functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that GF and FG are naturally isomorphic to the respective identities.

Proposition 6.2. *If a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between Waldhausen categories is exact, reflects weak equivalences and cofibrations, and is an equivalence of categories, then any inverse equivalence $G: \mathcal{D} \rightarrow \mathcal{C}$ is exact. Thus, F is a Waldhausen equivalence in the sense of Definition 6.1.*

Proof. Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is exact, reflects weak equivalences and cofibrations, and is an equivalence of categories, and $G: \mathcal{D} \rightarrow \mathcal{C}$ is a (not necessarily exact) functor equipped with two natural isomorphisms $GF \cong \text{Id}_{\mathcal{C}}$ and $FG \cong \text{Id}_{\mathcal{D}}$.

We have $* \cong GF(*) = G(*)$.

Suppose $n: d \rightarrow d'$ is a weak equivalence in \mathcal{D} . There exist c and c' in \mathcal{C} and isomorphisms $d \cong Fc$ and $d' \cong Fc'$ in \mathcal{D} , as well as a unique morphism $m: c \rightarrow c'$ in \mathcal{C} such that $F(m)$ is the composite

$$Fc \cong d \xrightarrow{n} d' \cong Fc'.$$

Then $F(m)$ is a weak equivalence, so m is also a weak equivalence. The natural isomorphism $GF \cong \text{Id}_{\mathcal{C}}$ then shows that $GF(m)$ is also a weak

equivalence. The morphism $G(n)$ is also a weak equivalence, as it is the composite

$$Gd \cong GFc \xrightarrow{GF(m)} GF(c') \cong Gd'$$

Note that we did not assume the 3-for-2 property for weak equivalences in order to prove G preserves weak equivalences, rather, we only used the axioms that weak equivalences are closed under composition and contain all isomorphisms. Since the class of cofibrations also satisfies these two axioms, the same proof shows that G preserves cofibrations.

The functor G sends pushouts along cofibrations to pushouts along cofibrations, as it is an equivalence.

Thus G is an exact functor (after replacing $G(*) = *'$ by $*$). \square

Proposition 6.3. *Every Waldhausen equivalence reflects weak equivalences and cofibrations.*

Proof. If F is a Waldhausen equivalence and G is an inverse Waldhausen equivalence, then the naturality diagram

$$(6.4) \quad \begin{array}{ccc} GF(c) & \xrightarrow{GF(m)} & GF(c') \\ \cong \downarrow & & \downarrow \cong \\ c & \xrightarrow{m} & c' \end{array}$$

shows that $F(m)$ is weak equivalence or cofibration if and only if m is. \square

Corollary 6.5. *If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a Waldhausen equivalence, then $wF: w\mathcal{C} \rightarrow w\mathcal{D}$ is an equivalence of categories.*

Proposition 6.6. *If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a Waldhausen equivalence and $n \geq 0$, then $wS_n F: wS_n \mathcal{C} \rightarrow wS_n \mathcal{D}$ is an equivalence of categories.*

Proof. The Waldhausen equivalence F induces a Waldhausen equivalence $S_n F$, to which Corollary 6.5 applies. \square

Corollary 6.7. *If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a Waldhausen equivalence, then*

$$wS_\bullet F: wS_\bullet \mathcal{C} \rightarrow wS_\bullet \mathcal{D}$$

is a weak equivalence of simplicial objects in \mathbf{Cat} . That is, the diagonal of the level-wise nerve is a weak equivalence of simplicial sets.

Proof. The Realization Lemma in combination with Proposition 6.6. \square

Having reached the goal for Waldhausen categories, we now turn to the quasicategorical analogues.

Definition 6.8. An exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between Waldhausen quasicategories is a *Waldhausen equivalence* if there is an exact functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that GF and FG are naturally equivalent to the respective identities.

Proposition 6.9. *If a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between Waldhausen quasicategories is exact, reflects cofibrations, and is an equivalence of quasicategories, then any inverse equivalence $G: \mathcal{D} \rightarrow \mathcal{C}$ is exact. Thus, F is a Waldhausen equivalence in the sense of Definition 6.8.*

Proof. Suppose $F: \mathcal{C} \rightarrow \mathcal{D}$ is exact, reflects cofibrations, and is an equivalence of quasicategories, and $G: \mathcal{D} \rightarrow \mathcal{C}$ is a (not necessarily exact) functor equipped with two natural equivalences $GF \simeq \text{Id}_{\mathcal{C}}$ and $FG \simeq \text{Id}_{\mathcal{D}}$. The functor F reflects equivalences in \mathcal{D} because F is an equivalence of quasicategories.

We have $* \simeq GF(*) = G(*)$.

Since G is a functor, it maps an equivalence in \mathcal{D} to an equivalence in \mathcal{C} , and by 1-fullness, $G(\mathcal{D}_{\text{equiv}}) \subseteq \mathcal{C}_{\text{equiv}}$.

Suppose $n: d \rightarrow d'$ is a cofibration in \mathcal{D} . We have a commutative square in \mathcal{D}

$$\begin{array}{ccc} FGd & \xrightarrow{FGn} & FGd' \\ \text{equiv} \downarrow & & \downarrow \text{equiv} \\ d & \xrightarrow{n} & d' \end{array}$$

so that FGn is a cofibration by the homotopy repleteness of $co\mathcal{D}$ in \mathcal{D} . But since F reflects cofibrations, Gn is also a cofibration. Moreover, $G(co\mathcal{D}) \subseteq co\mathcal{C}$ by an inductive argument using the 1-fullness of $co\mathcal{D}$ and $co\mathcal{C}$.

The functor G sends pushouts along cofibrations to pushouts along cofibrations, as it is an equivalence.

Thus G is an exact functor (after replacing $G(*) = *'$ by $*$). \square

Proposition 6.10. *Every Waldhausen equivalence between quasicategories reflects weak equivalences and cofibrations.*

Proof. Any equivalence of quasicategories reflects equivalences. Reflection of cofibrations follows from homotopy repleteness, compare with diagram (6.4). \square

Corollary 6.11. *If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a Waldhausen equivalence of quasicategories, then*

$$(S_{\bullet}^{\infty} F)_{\text{equiv}} : (S_{\bullet}^{\infty} \mathcal{C})_{\text{equiv}} \rightarrow (S_{\bullet}^{\infty} \mathcal{D})_{\text{equiv}}$$

is a diagonal weak equivalence of bisimplicial sets.

Proof. The functor $S_n^{\infty} F$ is an equivalence of quasicategories, since S_n^{∞} sends natural equivalences to natural equivalences. These same natural equivalences make $(S_n^{\infty} F)_{\text{equiv}}$ into an equivalence of categories, as their

components are equivalences in quasicategories. The result now follows from the Realization Lemma and the fact that every equivalence of quasicategories is a weak homotopy equivalence of simplicial sets. \square

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